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Construction of exact solutions of integral-functional equations is possible only in some cases, so vital question of finding and study conditions for the convergence of approximate methods for solving these equations and error estimates kolokation-iterative method.

Key words: *linear integral-functional equations, Fredholm's integral equation, inverse operator, approximate solution, collocational-iterative method.*

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HYPERBOLIC BOUNDARY VALUE PROBLEM FOR SEMIBOUNDED PIECEWISE-HOMOGENEOUS SOLID CYLINDER

By means of the method of integral and hybrid integral transforms, in combination with the method of main solutions (influence matrices and Green matrices) the integral image of exact analytical solution of hyperbolic boundary value problem of mathematical physics for semibounded piecewise-homogeneous solid cylinder is obtained for the first time.

Keywords: *hyperbolic equation, initial and boundary conditions, conjugate conditions, integral transforms, the main solutions.*

Introduction. The theory of hyperbolic boundary value problems for partial differential equations is an important section of the modern theory of differential equations which is intensively developing in the present time. The popularity of the problem is the consequence of the significance of its results in the development of many mathematical problems, as well as of its numerous applications in mathematical modeling of different processes and phenomenon of mechanics, physics, engineering, new technologies.

Significant results from the theory of Cauchy and initial-boundary value problems (mixed problems) for hyperbolic equations were obtained in the known works of J. Hadamard [1], L. Gording [2], Yu. Mitropolsky, G. Khoma, M. Hromyak [3], A. Samoilenko, B. Tkach [4], M. Smirnov [5], V. Chernyatyn [6] and others domestic and foreign mathematics.

It is well known that the complexity of a boundary-value problem significantly depends on the coefficients of equations (different types of degeneracy and features) and the geometry of domain (limited, unlimited, smoothness of the boundary, the presence of corner points, etc.) in which the problem is considered. The dependence of the properties of solutions of boundary value problems for linear, quasi-linear, and certain classes of nonlinear equations (elliptic, parabolic, hyperbolic) in homogeneous domains (homogeneous environments) on the above-mentioned properties of the coefficients of equations and geometry of domain are studied in detail, and functional spaces of correctness of problems in the sense of Hadamard are constructed.

However, many important applied problems of thermomechanics, thermal physics, diffusion, theory of elasticity, theory of electrical circuits, theory of vibrations of mechanical systems lead to boundary value problems and mixed problems not only in homogeneous domains when the coefficients of the equations are continuous, but also in inhomogeneous and piecewise homogeneous domains when the coefficients of the equations are piecewise continuous or piecewise constant [7, 8].

The method of hybrid integral transforms generated by hybrid differential operators when in each component of connectivity of piecewise homogeneous domain are treated different differential operators or differential operators look the same, but with different sets of coefficients is an effective method of constructing exact solutions for a fairly broad class of linear boundary value problems and mixed problems in piecewise homogeneous domains [9–12].

By means of the method of hybrid integral transforms the exact solution of hyperbolic boundary value problem of mathematical physics for semi-bounded piecewise homogeneous solid cylinder is obtained in this article.

Formulation of the problem. Let's consider the problem of structure of 2π-periodic for angular variable φ solution of partial differential equations of hyperbolic type of 2nd order [13]

$$\frac{\partial^2 u_j}{\partial t^2} - \left[a_{rj}^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) + \frac{a_{\varphi j}^2}{r^2}\frac{\partial^2}{\partial \varphi^2} + a_{zj}^2 \frac{\partial^2}{\partial z^2} \right] u_j + \\ + \chi_j^2 u_j = f_j(t, r, \varphi, z);\ r \in I_j;\ j = \overline{1, n+1} \tag{1}$$

which is bounded in the set

$$D=\{(t,r,\varphi,z): t>0; r\in I_n^+=\bigcup_{j=1}^{n+1} I_j\equiv\bigcup_{j=1}^{n+1}(R_{j-1};R_j),\ R_0\equiv 0, R_{n+1}\equiv R<+\infty;$$
$$\varphi\in[0;2\pi); z\in(0;+\infty)\}$$

with initial conditions

$$u_j\big|_{t=0}=g_j^1(r,\varphi,z);\ \ \frac{\partial u_j}{\partial t}\Big|_{t=0}=g_j^2(r,\varphi,z);\ \ r\in I_j;\ \ j=\overline{1,n+1} \tag{2}$$

boundary conditions

$$\left.\frac{\partial^s u_1}{\partial r^s}\right|_{r=0}=0;\ \left.\left(\alpha_{22}^{n+1}\frac{\partial}{\partial r}+\beta_{22}^{n+1}\right)u_{n+1}\right|_{r=R}=g(t,\varphi,z); \tag{3}$$

$$\left.\left(-\frac{\partial}{\partial z}+h\right)u_j\right|_{z=0}=g_j(t,r,\varphi);\ \left.\frac{\partial^s u_j}{\partial z^s}\right|_{z=+\infty}=0;\ \ s=0,1;\ j=\overline{1,n+1} \tag{4}$$

and conjugate conditions

$$\left.\left[\left(\alpha_{j1}^k\frac{\partial}{\partial r}+\beta_{j1}^k\right)u_k-\left(\alpha_{j2}^k\frac{\partial}{\partial r}+\beta_{j2}^k\right)u_{k+1}\right]\right|_{r=R_k}=0;\ \ j=1,2;\ k=\overline{1,n}, \tag{5}$$

here a_{rj}, $a_{\varphi j,}$ a_{zj}, χ_j, α_{js}^k, β_{js}^k — some not negative constants;

$$\alpha_{22}^{n+1}\ge 0;\ \beta_{22}^{n+1}\ge 0;\ \alpha_{22}^{n+1}+\beta_{22}^{n+1}\ne 0;$$
$$c_{jk}=\alpha_{2j}^k\beta_{1j}^k-\alpha_{1j}^k\beta_{2j}^k\ne 0;\ \ c_{1k}\cdot c_{2k}>0;$$
$$f(t,r,\varphi,z)=\{f_1(t,r,\varphi,z),f_2(t,r,\varphi,z),\dots,f_{n+1}(t,r,\varphi,z)\};$$
$$g^1(r,\varphi,z)=\left\{g_1^1(r,\varphi,z),g_2^1(r,\varphi,z),\dots,g_{n+1}^1(r,\varphi,z)\right\};$$
$$g^2(r,\varphi,z)=\left\{g_1^2(r,\varphi,z),g_2^2(r,\varphi,z),\dots,g_{n+1}^2(r,\varphi,z)\right\};$$
$$g(t,r,\varphi)=\{g_1(t,r,\varphi),g_2(t,r,\varphi),\dots,g_{n+1}(t,r,\varphi)\};\ g_0(t,\phi,z)$$

are known bounded continuous functions;

$$u(t,r,\varphi,z)=\{u_1(t,r,\varphi,z),u_2(t,r,\varphi,z),\dots,u_{n+1}(t,r,\varphi,z)\}$$

is the desired function.

The main part. Let's assume that the solution of the problem (1)–(5) exists and defined and the unknown functions satisfy the condition of applicability of direct and inverse integral and hybrid integral transformations (6)–(8) [14–16].

Let's apply the integral Fourier transform on Cartesian semiaxis $(0;+\infty)$ relative to variable z to the initial-boundary problem (1)–(5) [14]:

$$F_+\left[f(z)\right]=\int_{-\infty}^{+\infty} f(z)K(z,\sigma)dz\equiv\tilde{f}(\sigma), \tag{6}$$

$$F_+^{-1}\left[\tilde{f}(\sigma)\right] = \int_0^{+\infty} \tilde{f}(\sigma) K(z,\sigma) d\sigma = f(z), \tag{7}$$

$$F_+\left[\frac{d^2 f}{dz^2}\right] = -\sigma^2 \tilde{f}(\sigma) + K(0,\sigma)\left(-\frac{df}{dz} + hf\right)\Bigg|_{z=0}, \tag{8}$$

here conversion kernel is $K(z,\sigma) = \sqrt{\frac{2}{\pi}} \frac{\cos(\sigma z) + h\sin(\sigma z)}{\sqrt{\sigma^2 + h^2}}$.

The integral operator F_+ due to the formula (6) as a result of identity (8) three-dimensional initial boundary value problem of conjugation (1)–(5) puts in accordance the task of constructing solution which is limited in the set $D' = \left\{(t,r,\varphi); t > 0; r \in I_n^+; \varphi \in [0;2\pi)\right\}$ and is 2π-periodical of angular variable φ of differential equations

$$\begin{gathered}\frac{\partial^2 \tilde{u}_j}{\partial t^2} - \left[a_{rj}^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{a_{\varphi j}^2}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]\tilde{u}_j + \left(a_{zj}^2\sigma^2 + \chi_j^2\right)\tilde{u}_j = \\ = \tilde{F}_j(t,r,\varphi,\sigma);\ r \in I_j;\ j = \overline{1,n+1}\end{gathered} \tag{9}$$

with initial conditions

$$\tilde{u}_j\Big|_{t=0} = \tilde{g}_j^1(r,\varphi,\sigma);\ \ \frac{\partial \tilde{u}_j}{\partial t}\Bigg|_{t=0} = \tilde{g}_j^2(r,\varphi,\sigma);\ \ r \in I_j;\ j = \overline{1,n+1}; \tag{10}$$

boundary conditions

$$\frac{\partial^s \tilde{u}_1}{\partial r^s}\Bigg|_{r=0} = 0;\ \left(\alpha_{22}^{n+1}\frac{\partial}{\partial r} + \beta_{22}^{n+1}\right)\tilde{u}_{n+1}\Bigg|_{r=R} = \tilde{g}(t,\varphi,\sigma); \tag{11}$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^k\frac{\partial}{\partial r} + \beta_{j1}^k\right)\tilde{u}_k - \left(\alpha_{j2}^k\frac{\partial}{\partial r} + \beta_{j2}^k\right)\tilde{u}_{k+1}\right]\Bigg|_{r=R_k} = 0;\ \ j = 1,2;\ \ k = \overline{1,n}, \tag{12}$$

here $\tilde{F}_j(t,r,\varphi,\sigma) = \tilde{f}_j(t,r,\varphi,\sigma) + a_{zj}^2 K(0,\sigma) g_j(t,r,\varphi);\ \ j = \overline{1,n+1}.$

Let's apply finite integral Fourier transform relative to the variable φ to the problem (9)–(12) [15]:

$$F_m\left[g(\phi)\right] = \int_0^{2\pi} g(\varphi)\exp(-im\varphi)d\varphi \equiv g_m,\ \ i = \sqrt{-1}; \tag{13}$$

$$F_m^{-1}[g_m] = \frac{\mathrm{Re}}{2\pi}\sum_{m=0}^{\infty} \varepsilon_m g_m \exp(im\varphi) \equiv g(\varphi), \tag{14}$$

$$F_m\left[\frac{d^2g}{d\varphi^2}\right]=-m^2F_m[g(\varphi)]\equiv -m^2g_{m,} \tag{15}$$

here $\mathrm{Re}(\cdots)$ — the real part of the expression $(\cdots)$ relative to the variable φ; $\varepsilon_0=1$, $\varepsilon_k=2$; $k=1,2,3\ldots$

The integral operator F_m due to the formula (13) as a result of identity (15) two-dimensional initial boundary value problem of conjugation (9)–(12) puts in accordance the task of constructing solution which is limited in the set $D''=\left\{(t,r);t>0;r\in I_n^+\right\}$ of separate system of one-dimensional hyperbolic differential equations of 2nd order

$$\frac{\partial^2\tilde{u}_{jm}}{\partial t^2}-a_{rj}^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\nu_{jm}^2}{r^2}\right)\tilde{u}_{jm}+\left(a_{zj}^2\sigma^2+\chi_j^2\right)\tilde{u}_{jm}= \\ =\tilde{F}_{jm}(t,r,\sigma);\ r\in I_j;\ j=\overline{1,n+1};\ \nu_{jm}=a_{\varphi j}m/a_{rj} \tag{16}$$

with initial conditions

$$\tilde{u}_{jm}\Big|_{t=0}=\tilde{g}_{jm}^1(r,\sigma);\quad \frac{\partial\tilde{u}_{jm}}{\partial t}\Bigg|_{t=0}=\tilde{g}_{jm}^2(r,\sigma);\quad r\in I_j;\ j=\overline{1,n+1}, \tag{17}$$

boundary conditions

$$\frac{\partial^s\tilde{u}_{1m}}{\partial r^s}\Bigg|_{r=0}=0;\quad \left(\alpha_{22}^{n+1}\frac{\partial}{\partial r}+\beta_{22}^{n+1}\right)\tilde{u}_{n+1,m}\Bigg|_{r=R}=\tilde{g}_m(t,\sigma); \tag{18}$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^k\frac{\partial}{\partial r}+\beta_{j1}^k\right)\tilde{u}_{km}-\left(\alpha_{j2}^k\frac{\partial}{\partial r}+\beta_{j2}^k\right)\tilde{u}_{k+1,m}\right]\Bigg|_{r=R_k}=0;\quad j=1,2;\ k=\overline{1,n}. \tag{19}$$

Let's apply finite hybrid integral Hankel transform of 1st kind relative to the variable r in piecewise homogeneous segment I_n^+ of n conjugation points to the problem (16)–(19) [16]:

$$H_{sn}\left[f(r)\right]=\int_0^R f(r)V(r,\lambda_s)\sigma(r)rdr\equiv\tilde{f}(\lambda_s), \tag{20}$$

$$H_{sn}^{-1}\left[\tilde{f}(\lambda_s)\right]=\sum_{s=1}^{\infty}\tilde{f}(\lambda_s)\frac{V(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}\equiv f(r), \tag{21}$$

$$H_{sn}\left[B_{(m)}[f(r)]\right]=-\lambda_s^2\tilde{f}(\lambda_s)-\sum_{k=1}^{n+1}\gamma_k^2\int_{R_{k-1}}^{R_k}f(r)V_k(r,\lambda_s)\sigma_k rdr+ \\ +a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\left(\alpha_{22}^{n+1}\frac{df}{dr}+\beta_{22}^{n+1}f\right)\Bigg|_{r=R};\ R_0=0;\ R_{n+1}=R. \tag{22}$$

Spectral function $V(r,\lambda_s)$, weight function $\sigma(r)$ and hybrid Bessel differential operator $B_{(m)} = \sum_{k=1}^{n+1} a_{rk}^2 \theta(r - R_{k-1})\theta(R_k - r) B_{\nu_{km}}$, written in [16], take part in formulas (20)–(22).

Here $B_{\nu_{km}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu_{km}^2}{r^2}$ is classical Bessel differential operator, $\theta(x)$ is the Heaviside step function.

Let's write the separate system of differential equations (16) and the initial conditions (17) in matrix form

$$\begin{bmatrix} \left(\frac{\partial^2}{\partial t^2} - a_{r1}^2 B_{\nu_{1m}} + q_1^2(\sigma)\right)\tilde{u}_{1m} \\ \left(\frac{\partial^2}{\partial t^2} - a_{r2}^2 B_{\nu_{2m}} + q_2^2(\sigma)\right)\tilde{u}_{2m} \\ \cdots\cdots\cdots \\ \left(\frac{\partial^2}{\partial t^2} - a_{r,n+1}^2 B_{\nu_{n+1,m}} + q_{n+1}^2(\sigma)\right)\tilde{u}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{1m}(t,r,\sigma) \\ \tilde{F}_{2m}(t,r,\sigma) \\ \cdots\cdots \\ \tilde{F}_{n+1,m}(t,r,\sigma) \end{bmatrix}, \qquad (23)$$

$$\left.\begin{bmatrix} \tilde{u}_{1m}(t,r,\sigma) \\ \tilde{u}_{2m}(t,r,\sigma) \\ \cdots\cdots \\ \tilde{u}_{n+1,m}(t,r,\sigma) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \tilde{g}_{1m}^1(r,\sigma) \\ \tilde{g}_{2m}^1(r,\sigma) \\ \cdots\cdots \\ \tilde{g}_{n+1,m}^1(r,\sigma) \end{bmatrix};$$

$$\frac{\partial}{\partial t}\left.\begin{bmatrix} \tilde{u}_{1m}(t,r,\sigma) \\ \tilde{u}_{2m}(t,r,\sigma) \\ \cdots\cdots \\ \tilde{u}_{n+1,m}(t,r,\sigma) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \tilde{g}_{1m}^2(r,\sigma) \\ \tilde{g}_{2m}^2(r,\sigma) \\ \cdots\cdots \\ \tilde{g}_{n+1,m}^2(r,\sigma) \end{bmatrix}, \qquad (24)$$

here $q_j^2(\sigma) = a_{zj}^2\sigma^2 + \chi_j^2;\ \ j = \overline{1, n+1}.$

Let's represente the integral operator H_{sn} which operates due to the formula (20) as an operator matrix-row:

$$H_{sn}[\cdots] = \left[\int_0^{R_1} \cdots V_1(r,\lambda_s)\sigma_1 r dr \quad \int_{R_1}^{R_2} \cdots V_2(r,\lambda_s)\sigma_2 r dr \right.$$

$$\left. \int_{R_{n-1}}^{R_n} \cdots V_n(r,\lambda_s)\sigma_n r dr \quad \int_{R_n}^{R} \cdots V_{n+1}(r,\lambda_s)\sigma_{n+1} r dr \right] \qquad (25)$$

Let's apply the operator matrix-row (25) to the problem (23), (24) according to the matrix multiplication rule. As a result of the identity (22), we get a Cauchy problem

$$\sum_{j=1}^{n+1}\left(\frac{d^2}{dt^2}+\lambda_s^2+\gamma_j^2+q_j^2(\sigma)\right)\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma)= \\ =\sum_{j=1}^{n+1}\tilde{F}_{jm}(t,\lambda_s,\sigma)+a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(t,\sigma), \quad (26)$$

$$\left.\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma)\right|_{t=0}=\sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jm}^1(\lambda_s,\sigma); \\ \left.\frac{d}{dt}\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma)\right|_{t=0}=\sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jm}^2(\lambda_s,\sigma), \quad (27)$$

де $\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma)=\int\limits_{R_{j-1}}^{R_j}\tilde{u}_{jm}(t,r,\sigma)V_j(r,\lambda_s)\sigma_j r dr;\quad j=\overline{1,n+1},$

$$\tilde{\tilde{F}}_{jm}(t,\lambda_s,\sigma)=\int\limits_{R_{j-1}}^{R_j}\tilde{F}_{jm}(t,r,\sigma)V_j(r,\lambda_s)\sigma_j r dr,\quad j=\overline{1,n+1},$$

$$\tilde{\tilde{g}}_{jm}^k(\lambda_s,\sigma)=\int\limits_{R_{j-1}}^{R_j}\tilde{g}_{jm}^k(r,\sigma)V_j(r,\lambda_s)\sigma_j r dr;\quad k=1,2;\quad j=\overline{1,n+1}.$$

Let's suppose that $\max\left\{q_1^2,q_2^2,...,q_{n+1}^2\right\}=q_1^2$ and put everywhere $\gamma_j^2=q_1^2-q_j^2;\ j=\overline{1,n+1}$. Cauchy problem (26), (27) takes the form

$$\frac{d^2\tilde{\tilde{u}}_m}{dt^2}+\Delta^2(\lambda_s,\sigma)\tilde{\tilde{u}}_m=\tilde{\tilde{F}}_m(t,\lambda_s,\sigma)+ \\ +a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(t,\sigma), \quad (28)$$

$$\left.\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)\right|_{t=0}=\tilde{\tilde{g}}_m^1(\lambda_s,\sigma);\quad \left.\frac{d\tilde{\tilde{u}}_m}{dt}\right|_{t=0}=\tilde{\tilde{g}}_m^2(\lambda_s,\sigma), \quad (29)$$

де $\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)=\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma);\quad \tilde{\tilde{F}}_m(t,\lambda_s,\sigma)=\sum_{j=1}^{n+1}\tilde{\tilde{F}}_{jm}(t,\lambda_s,\sigma),$

$$\tilde{\tilde{g}}_m^k(\lambda,\sigma)=\sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jm}^k(\lambda_s,\sigma);\quad k=1,2;\quad \Delta^2(\lambda_s,\sigma)=\lambda_s^2+a_{z1}^2\sigma^2+\chi_1^2.$$

It is directly verify that the only solution of the inhomogeneous Cauchy problem (28), (29) is a function

$$\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)=G(t,\lambda_s,\sigma)\tilde{\tilde{g}}_m^2(\lambda_s,\sigma)+\frac{d}{dt}G(t,\lambda_s,\sigma)\tilde{\tilde{g}}_m^1(\lambda_s,\sigma)+$$
$$+\int_0^t G(t-\tau,\lambda_s,\sigma)\left[\tilde{\tilde{F}}_m(\tau,\lambda_s,\sigma)+a_{n+1}^2 R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(t,\sigma)\right]d\tau, \tag{30}$$

here Cauchy function $G(t,\lambda_s,\sigma)=\dfrac{\sin(\Delta(\lambda_s,\sigma)t)}{\Delta(\lambda_s,\sigma)}$.

Integral operator H_{sn}^{-1}, as inverse to H_{sn}, we represent as the operator matrix-column:

$$H_{sn}^{-1}[\cdots]=\begin{bmatrix}\sum\limits_{s=1}^{\infty}\cdots\dfrac{V_1(r,\lambda_s)}{\|V(r,\lambda_s)\|^2}\\ \sum\limits_{s=1}^{\infty}\cdots\dfrac{V_2(r,\lambda_s)}{\|V(r,\lambda_s)\|^2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \sum\limits_{s=1}^{\infty}\cdots\dfrac{V_{n+1}(r,\lambda_s)}{\|V(r,\lambda_s)\|^2}\end{bmatrix} \tag{31}$$

Let's apply operator matrix-column (31) to the matrix-element $\left[\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)\right]$, where the function $\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)$ is defined by formula (30) due to matrices multiplication rule. As a result we get the only solution of one-dimensional hyperbolic initial boundary problem of conjugation (16)–(19):

$$\tilde{u}_{jm}(t,r,\sigma)=\sum_{s=1}^{\infty}\left[G(t,\lambda_s,\sigma)\tilde{\tilde{g}}_m^2(\lambda_s,\sigma)+\frac{\partial}{\partial t}G(t,\lambda_s,\sigma)\tilde{\tilde{g}}_m^1(\lambda_s,\sigma)\right]\times$$
$$\times\frac{V_j(r,\lambda_s)}{\|V(r,\lambda_s)\|^2}+\sum_{j=1}^{\infty}\int_0^t G(t-\tau,\lambda_s,\sigma)\left[\tilde{\tilde{F}}_m(\tau,\lambda_s,\sigma)+\right. \tag{32}$$
$$\left.+a_{n+1}^2 R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(\tau,\sigma)\right]d\tau\frac{V_j(r,\lambda_s)}{\left\|V\left(r,\lambda_s\right)\right\|^2}.$$

If to apply consistently inverse operators F_+^{-1} and F_m^{-1} to functions $\tilde{u}_{jm}(t,r,\sigma)$, which are defined by formulas (32) and perform the some simple transformation, we get functions

$$u_j(t,r,\varphi,z)=\sum_{k=1}^{n+1}\int_0^t\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\int_0^{+\infty}E_{jk}(t-\tau,r,\rho,\varphi-\alpha,z,\xi)f_k(\tau,\rho,\alpha,\xi)\times$$

$$\times\sigma_k\rho d\xi d\alpha d\rho d\tau+\frac{\partial}{\partial t}\sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\int_0^{+\infty}E_{jk}(t,r,\rho,\varphi-\alpha,z,\xi)g_k^1(\rho,\alpha,\xi)\times$$

$$\times\sigma_k\rho d\xi d\alpha d\rho+\sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\int_0^{+\infty}E_{jk}(t,r,\rho,\varphi-\alpha,z,\xi)g_k^2(\rho,\alpha,\xi)\times \quad (33)$$

$$\times\sigma_k\rho d\xi d\alpha d\rho+\sum_{k=1}^{n+1}a_{zk}^2\int_0^t\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\left[W_{jk}\left(t-\tau,r,\rho,\varphi-\alpha,z\right)g_k(\tau,\rho,\alpha)\times\right.$$

$$\times\sigma_k\rho d\alpha d\rho d\tau+\int_0^t\int_0^{2\pi}\int_0^{+\infty}W_{jr}\left(t-\tau,r,\varphi-\alpha,z,\xi\right)g(\tau,\alpha,\xi)d\xi d\alpha d\tau;\ j=\overline{1,n+1},$$

Functions (33) define the only solution of hyperbolic initial boundary problem of conjugation (1)–(5).

In formulas (33) there are components

$$E_{jk}(t,r,\rho,\varphi,z,\xi)=\frac{1}{2\pi}\sum_{m=0}^{\infty}\varepsilon_m\sum_{s=1}^{\infty}\int_0^{+\infty}G(t,\lambda_s,\sigma)K(z,\sigma)K(\xi,\sigma)d\sigma\times$$

$$\times\frac{V_j(r,\lambda_s)V_k(\rho,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}\cos(m\varphi);\ j,k=\overline{1,n+1}$$

of matrix of influence (function of influence), components $W_{jk}(t,r,\rho,\varphi,z)=E_{jk}(t,r,\rho,\varphi,z,0)$ of tangential Green's matrix (tangential Green's function) and components $W_{jr}(t,r,\varphi,z,\xi)=$ $=a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}E_{j,n+1}(t,r,R,\varphi,z,\xi)$ of radial Green's matrix (radial Green's function) of considered problem.

Using a properties of functions of influence $E_{jk}(t,r,\rho,\varphi,z,\xi)$ and Green's functions $W_{jk}(t,r,\rho,\varphi,z)$, $W_{jr}(t,r,\varphi,z,\xi)$ we can verify that functions $u_j(t,r,\varphi,z)$ which are defined by formulas (33) satisfy the equation (1), the initial conditions (2), the boundary conditions (3), (4) and conjugate conditions (5) in the sense of theory of generalized functions [17].

The uniqueness of the solution (33) follows from its structure (integrated image) and from uniqueness of the main solutions (functions of influence and Green's functions) of problem (1)–(5).

By methods from [17, 18] can be proved that under appropriate conditions on the initial data, formulas (33) define a limited classical solution of the hyperbolic initial boundary problem of conjugation (1)–(5).

We get the following theorem as the summary of the above results.

Theorem. If functions $f_j(t,r,\varphi,z)$, $g_j^1(r,\varphi,z)$, $g_j^2(r,\varphi,z)$, $g_j(t,r,\varphi)$ satisfy conditions:

1) are continuously differentiated twice for each variable;
2) have a limited variation for the geometric variables;
3) are absolutely summable with the variable z in $(0;+\infty)$;
4) conjugate conditions are true and function $g(t,\varphi,z)$ is continuously differentiated twice for each variable, has a limited variation for the geometric variables, is absolutely summable with the variable z in $(0;+\infty)$, then hyperbolic initial boundary value problem (1)–(5) has the only limited classical solution, which is determined by formula (33).

Remark 1. In the case of $a_{rj} = a_{\varphi j} = a_{zj} \equiv a_j > 0$ formulas (33) define the structure of the solution of hyperbolic initial boundary value problem (1)–(5) in an isotropic semibounded piecewise homogeneous solid cylinder.

Remark 2. Parameters α_{22}^{n+1}, β_{22}^{n+1} allow to allocate the solutions of initial boundary value problems from formulas (33) in the case of boundary condition of the 1st kind ($\alpha_{22}^{n+1} = 0,\ \beta_{22}^{n+1} = 1$), 2nd kind ($\alpha_{22}^{n+1} = 1,\ \beta_{22}^{n+1} = 0$) and 3rd kind ($\alpha_{22}^{n+1} = 1,\ \beta_{22}^{n+1} \equiv \beta > 0$) on the radial surface $r = R$.

Remark 3. Parameter h allow to allocate the solutions of initial boundary value problems from formulas (33) in the case of boundary condition of the 1st kind ($h \to \infty$) and 2nd kind ($h \to 0$) on the surface $z = 0$.

Remark 4. Analysis of the solution (33) is done directly from the general structure according to the analytical expression of functions $f_j(t,r,\varphi,z)$, $g_j^1(r,\varphi,z)$, $g_j^2(r,\varphi,z)$ $g_j(t,r,\varphi)$, $g(t,\varphi,z)$.

Remark 5. In the case of $\chi_j \equiv 0$ equation (1) is a classic three-dimensional inhomogeneous wave equation (the equation of fluctuations) for an orthotropic environment in cylindrical coordinates.

Remark 6. In the case of $\alpha_{11}^k = 0,\ \beta_{11}^k = 1;\ \alpha_{12}^k = 0,\ \beta_{12}^k = 1;$ $\alpha_{21}^k = E_1^k,\ \beta_{21}^k = 0;\ \alpha_{22}^k = E_2^k,\ \beta_{22}^k = 0,$ here E_1^k, E_2^k — Young's modulus ($k = \overline{1,n},$), the conjugate conditions (5) coincide with conditions of ideal mechanical contact.

Thus, in these cases 5, 6 considered hyperbolic boundary value problem of mathematical physics (1)–(5) is a mathematical model of forced oscillating processes in semibounded piecewise homogeneous solid cylinder.

Conclusions. By means of method of integral and hybrid integral transforms with the method of principal solutions (influence functions and Green’s functions) integral image of exact analytical solution of hyperbolic boundary-value problem of mathematical physics in semibounded piecewise homogeneous solid cylinder is obtained. The obtained solution is of algorithmic character, continuously depend on the parameters and data of problem and can be used in further theoretical research and in practical engineering calculations of real processes which are modeled by hyperbolic boundary-value problems of mathematical physics in piecewise homogeneous domains.

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Методом інтегральних і гібридних інтегральних перетворень у поєднанні з методом головних розв'язків (матриць впливу та матриць Гріна) вперше побудовано інтегральне зображення єдиного точного аналітичного розв'язку гіперболічної крайової задачі математичної фізики для напівобмеженого кусково-однорідного суцільного циліндра.

Ключові слова: *гіперболічне рівняння, початкові та крайові умови, умови спряження, інтегральні перетворення, головні розв'язки.*

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КРИТЕРІЇ ЕКСТРЕМАЛЬНОЇ ПОСЛІДОВНОСТІ ДЛЯ ЗАДАЧІ НАЙКРАЩОГО У РОЗУМІННІ ОПУКЛОЇ ФУНКЦІЇ НАБЛИЖЕННЯ ФІКСОВАНОГО ЕЛЕМЕНТА ОПУКЛОЮ МНОЖИНОЮ

У статті встановлено критерії екстремальної послідовності для задачі найкращого у розумінні опуклої функції наближення фіксованого елемента лінійного нормованого простору опуклою множиною цього простору.

Ключові слова: *опукла функція, опукла множина, задача найкращого наближення, екстремальна послідовність, критерій екстремальної послідовності.*

Вступ. У статті для задачі найкращого у розумінні опуклої функції наближення фіксованого елемента лінійного нормованого простору опуклою множиною цього простору встановлено критерії екстремальної послідовності, окремі з яких узагальнюють критерії екстремального елемента для цієї задачі, встановлені у праці [1].

Постановка задачі. Нехай X — дійсний лінійний нормований простір, F — опукла множина простору X, p — опукла та неперервна на X функція, x — елемент простору X.

Задачею найкращого у розумінні функції p наближення елемента x множиною F будемо називати задачу відшукання величини