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PARABOLIC BOUNDARY VALUE PROBLEMS IN UNLIMITED PIECEWISE HOMOGENEOUS WEDGE-SHAPED HOLLOW CYLINDER

The unique exact analytical solutions of parabolic boundary value problems of mathematical physics in unbounded by variable z piecewise-homogeneous by radially variable r wedge-shaped by an angularly variable φ hollow cylinder were constructed at first time by the method of classical integral and hybrid integral transforms in combination with the method of main solutions (matrices of influence and Green matrices) in the proposed article.

The cases of assigning on the verge of the wedge the boundary conditions of Dirichlet and Neumann and their possible combinations (Dirichlet — Neumann, Neumann — Dirichlet) are considered.

Finite integral Fourier transform by an angular variable, a Fourier integral transform on a Cartesian axis by an applicative variable and a hybrid integral transform of the Hankel type of the second kind on a segment of the polar axis with n points of conjugation were used to construct classic solutions of investigated initial-boundary value problems.

The consistent application of integral transforms by geometric variables allows us to reduce the three-dimensional initial boundary-value problems of conjugation to the Cauchy problem for a regular linear inhomogeneous 1st order differential equation whose unique solution is written in a closed form.

The application of inverse integral transforms restores explicitly the solution of the considered problems through their integral image.

Key words: *parabolic equation, initial and boundary conditions, conjugation conditions, integral transforms, hybrid integral transforms, main solutions.*

Introduction. The theory of boundary value problems for partial differential equations and the equations of mathematical physics in particular, is an important part of modern theory of differential equations, which is developing intensively in our time. Its results are important for the development of many branches of mathematics, and numerous applications of its achievements are importance for study of various mathematical models of various processes and phenomena of physics, mechanics, chemistry, biology, medicine, economics, ecology, technology, latest technologies.

Significant results from the theory of the Cauchy problem and boundary value problems for equations of parabolic type were obtained in the well-known works of V. Gorodetsky [2], Zhitarashu N., Eidelman S. [6], Zagorskiy T. [7], Ivasishen S. [8], Ladyzhenskaya O., Solonnikova V., Ural’ceva N. [14], Landis E. [15], Matiychuk M. [16], Pukalskyi I. [18], Friedman A. [22], Eidelman S. [24] and other domestic and foreign mathematicians.

It is well known that the complexity of the studied boundary value problems significantly depends on the properties of the coefficients of the equations (different types of degeneracies and features of the coefficients) and on the geometric structure of the region (smoothness of the boundary, angular points, boundedness, infinity, etc.). At present, the properties of solutions have been studied in detail and various methods for constructing solutions (exact and approximate) of boundary value problems for linear, quasilinear, and some nonlinear equations of different types (elliptic, parabolic, hyperbolic) in single-connected domains (homogeneous media) have been developed and functional spaces of correctness of problems in the sense of Hadamard have been constructed.

However, many important applied problems of thermomechanics, thermal physics, diffusion, theory of elasticity, theory of electrical circuits, oscillation theory, mechanics of a deformable solid lead to boundary value problems and mixed problems not only in homogeneous environments when the coefficients of the equations are continuous, but also in inhomogeneous and piecewise homogeneous environments when the coefficients of equations are piecewise continuous or piecewise constant [4, 5, 19].

It is known that in addition to the method of separation of variables (Fourier method) and its generalizations, one of the important and effective methods of studying linear boundary and mixed problems for partial differential equations in homogeneous environments is the method of integral transforms, which allows to construct analytically exact solutions of the considered problems through their integral image.

At the same time, for a rather wide class of linear boundary value problems in piecewise homogeneous environments, the method of hybrid integral transforms generated by the corresponding hybrid differential operators on each component of connectivity of piecewise homogeneous environment wiht different differential operators, or differential operators of the same type, but with different sets of coefficients proved to be an effective method of constructing their solutions [3, 9-12].

This article is a logical continuation of [13]. Integral images of the only exact analytical solutions of parabolic initial-boundary value problems of mathematical physics in an unbounded piecewise-homogeneous wedge-shaped hollow cylinder were constructed in this article by means of the method of integral and hybrid integral transforms in combination with the method of principal solutions.

Formulation of the problem. Let's consider the problem of constructing a classical solution of linear partial differential equations of parabolic type of the 2nd order [17]

$$\frac{\partial u_j}{\partial t} - \left[a_{rj}^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) + \frac{a_{\varphi j}^2}{r^2}\frac{\partial^2}{\partial \varphi^2} + a_{zj}^2 \frac{\partial^2}{\partial z^2} \right] u_j + \tag{1}$$

$$+\chi_j^2 u_j = f_j(t,r,\varphi,z);\ r \in I_j;\ j = \overline{1,n+1},$$

which is bounded in the set

$$D = \{(t,r,\varphi,z) : t > 0; r \in I_n^+ = \bigcup_{j=1}^{n+1} I_j \equiv \bigcup_{j=1}^{n+1} (R_{j-1}; R_j), R_0 > 0, R_{n+1} = R < +\infty;$$

$$\varphi \in (0;\varphi_0), 0 < \varphi_0 < 2\pi; z \in (-\infty;+\infty)\}$$

with initial conditions

$$u_j(t,r,\varphi,z)\big|_{t=0} = g_j(r,\varphi,z);\ \ r \in I_j;\ \ j = \overline{1,n+1}, \tag{2}$$

boundary conditions

$$\left.\frac{\partial^s u_j}{\partial z^s}\right|_{z=-\infty} = 0;\ \ \left.\frac{\partial^s u_j}{\partial z^s}\right|_{z=+\infty} = 0;\ \ s = 0,1;\ \ j = \overline{1,n+1}, \tag{3}$$

$$\left.\left(\alpha_{11}^0 \frac{\partial}{\partial r} + \beta_{11}^0\right) u_1\right|_{r=R_0} = g_0(t,\varphi,z);\ \ \left.\left(\alpha_{22}^{n+1} \frac{\partial}{\partial r} + \beta_{22}^{n+1}\right) u_{n+1}\right|_{r=R} = g(t,\varphi,z), \tag{4}$$

one of the boundary conditions on the wedge boundaries [13]

$$u_j\big|_{\varphi=0} = g_{1j}(t,r,z);\ \ u_j\big|_{\varphi=\varphi_0} = w_{1j}(t,r,z);\ \ j = \overline{1,n+1}, \tag{5}$$

$$u_j\big|_{\varphi=0} = g_{2j}(t,r,z);\ \ \left.\frac{\partial u_j}{\partial \varphi}\right|_{\varphi=\varphi_0} = -w_{2j}(t,r,z);\ \ j = \overline{1,n+1}, \tag{6}$$

$$\left.\frac{\partial u_j}{\partial \varphi}\right|_{\varphi=0} = g_{3j}(t,r,z);\ \ u_j\big|_{\varphi=\varphi_0} = w_{3j}(t,r,z);\ \ j = \overline{1,n+1}, \tag{7}$$

$$\left.\frac{\partial u_j}{\partial \varphi}\right|_{\varphi=0} = g_{4j}(t,r,z);\ \ \left.\frac{\partial u_j}{\partial \varphi}\right|_{\varphi=\varphi_0} = -w_{4j}(t,r,z);\ \ j = \overline{1,n+1} \tag{8}$$

and conjugate conditions [12]

$$\left.\left[\left(\alpha_{j1}^k \frac{\partial}{\partial r} + \beta_{j1}^k\right) u_k - \left(\alpha_{j2}^k \frac{\partial}{\partial r} + \beta_{j2}^k\right) u_{k+1}\right]\right|_{r=R_k} = 0;\ j = 1,2;\ k = \overline{1,n}, \tag{9}$$

here a_{rj}, $a_{\varphi j,}$ a_{zj}, χ_j, α_{js}^k, β_{js}^k — some constants;

$$c_{jk} = \alpha_{2j}^k \beta_{1j}^k - \alpha_{1j}^k \beta_{2j}^k \neq 0;\ \ c_{1k} \cdot c_{2k} > 0;\ \ \alpha_{11}^0 \leq 0,\ \ \beta_{11}^0 \geq 0,\ \alpha_{22}^{n+1} \geq 0,$$

$$\beta_{22}^{n+1} \geq 0;\ \ \left|\alpha_{11}^0\right| + \beta_{11}^0 \neq 0;\ \alpha_{22}^{n+1} + \beta_{22}^{n+1} \neq 0;$$

$$f(t,r,\varphi,z) = \{ f_1(t,r,\varphi,z),\ f_2(t,r,\varphi,z), \dots,\ f_{n+1}(t,r,\varphi,z)\};$$

$$g(r,\varphi,z) = \{g_1(r,\varphi,z), g_2(r,\varphi,z), \dots, g_{n+1}(r,\varphi,z)\};\ \ g_0(t,\varphi,z),$$

$$g(t,\varphi,z),\ \ g_{pj}(t,r,z),\ \ w_{pj}(t,r,z);\ \ (p = \overline{1,4};\ j = \overline{1,n+1})$$

— are known real bounded continuous functions;

$$u(t,r,\varphi,z) = \{u_1(t,r,\varphi,z), u_2(t,r,\varphi,z), \dots, u_{n+1}(t,r,\varphi,z)\}$$

— is desired real function wich is continuously differentiable by variable t and twice-continuously differentiable by geometric variables (r,φ,z).

Let's notice that:

1) in the case of $\chi_j \equiv 0$ ($j = \overline{1,n+1}$) equation (1) is a classic three-dimensional inhomogeneous thermal conductivity equation (diffusion) for an orthotropic environment in cylindrical coordinates;
2) in the case of $\alpha_{11}^k = 0$, $\beta_{11}^k = 1$; $\alpha_{12}^k = 0$, $\beta_{12}^k = 1$; $\alpha_{21}^k = \lambda_1^k$, $\beta_{21}^k = 0$; $\alpha_{22}^k = \lambda_2^k$, $\beta_{22}^k = 0$, here λ_1^k, λ_2^k — thermal conductivity coefficients, the conjugate conditions (9) coincide with conditions of ideal heat (thermal) contact;
3) in the case of $\alpha_{11}^k = b_k$, $\beta_{11}^k = 1$; $\alpha_{12}^k = 0$, $\beta_{12}^k = 1$; $\alpha_{21}^k = \lambda_1^k$, $\beta_{21}^k = 0$; $\alpha_{22}^k = \lambda_2^k$, $\beta_{22}^k = 0$, here b_k — coefficients of thermal resistance, the conjugate conditions (9) coincide with conditions of not ideal thermal contact.

Thus, in these cases 1, 2 (or 1, 3) considered parabolic boundary value problem of mathematical physics is a mathematical model of thermal conductivity processes in an unlimited piecewise homogeneous wedge-shaped hollow cylinder.

The main part. Let's assume that the solutions of parabolic initial-boundary problems of conjugation (1)-(4), (5), (9); (1)-(4), (6), (9); (1)-(4), (7), (9); (1)-(4), (8), (9) exist, and defined and the unknown functions satisfy the conditions of applicability of direct and inverse integral and hybrid integral transforms [12, 20, 21].

Due to [21] let's define finite direct $F_{m,ik}$ and inverse $F_{m,ik}^{-1}$ integral Fourier transforms relative to the angular variable $\varphi \in (0;\varphi_0)$ by formulas:

$$F_{m,ik}[f(\varphi)] = \int_0^{\varphi_0} f(\varphi) U_{m,ik}(\varphi) d\varphi \equiv f_{m,ik}, \tag{10}$$

$$F_{m,ik}^{-1}[f_{m,ik}] = \frac{2}{\varphi_0}\sum_{m=0}^{\infty}\varepsilon_m^{ik} f_{m,ik} U_{m,ik}(\varphi) \equiv f(\varphi), \tag{11}$$

here

$$U_{m,11}(\varphi) = \sin(\beta_{m,11}\varphi);\quad \beta_{m,11} = \frac{\pi m}{\varphi_0};$$

$$U_{m,12}(\varphi) = \sin(\beta_{m,12}\varphi);\quad \beta_{m,12} = \frac{\pi(2m+1)}{2\varphi_0};$$

$$U_{m,21}(\varphi) = \cos(\beta_{m,21}\varphi);\quad \beta_{m,21} = \beta_{m,12};$$

$$U_{m,22}(\varphi) = \cos(\beta_{m,22}\varphi);\quad \beta_{m,22} = \beta_{m,11};$$

$$\varepsilon_0^{ik} = 0;\quad \varepsilon_m^{ik} = 1 \ \text{if}\ ik = 11, 12, 21;\quad m = 1, 2, 3, \ldots;$$

$$\varepsilon_0^{22} = \frac{1}{2};\quad \varepsilon_m^{22} = 1 \ \text{if}\ m = 1, 2, 3, \ldots.$$

In this case, the identity (12) is fulfilled for the integral operator

$$F_{m,ik}\left[\frac{d^2 f}{d\varphi^2}\right] = -\beta_{m,ik}^2 f_{m,ik} + \Phi_{m,ik};\quad i,k = 1, 2, \tag{12}$$

here

$$\Phi_{m,11}(f) = \frac{\pi m}{\varphi_0}\left[f(0) + (-1)^{m+1} f(\varphi_0)\right];$$

$$\Phi_{m,12}(f) = \frac{\pi(2m+1)}{2\varphi_0} f(0) + (-1)^m \left.\frac{df}{d\varphi}\right|_{\varphi=\varphi_0};$$

$$\Phi_{m,21}(f) = -\left.\frac{df}{d\varphi}\right|_{\varphi=0} + (-1)^m \frac{\pi(2m+1)}{2\varphi_0} f(\varphi_0);$$

$$\Phi_{m,22} = -\left.\frac{df}{d\varphi}\right|_{\varphi=0} + (-1)^m \left.\frac{df}{d\varphi}\right|_{\varphi=\varphi_0}.$$

The integral operator $F_{m,ik}$ due to the formula (10) as a result of identity (12) three-dimensional initial boundary value problems of conjugation (1)-(4), (5), (9); (1)-(4), (6), (9); (1)-(4), (7), (9); (1)-(4), (8), (9) puts in accordance the task of constructing classical solution of two-dimensional differential equations of parabolic type of the 2nd order which is limited in the set $D' = \left\{(t, r, z) : t > 0; r \in I_n^+; z \in (-\infty; +\infty)\right\}$

$$\frac{\partial u_{jm,ik}}{\partial t} - \left[a_{rj}^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu_{jm,ik}^2}{r^2}\right) + a_{zj}^2\frac{\partial^2}{\partial z^2}\right]u_{jm,ik} + \chi_j^2 u_{jm,ik} = \tag{13}$$

$$= G_{jm,ik}(t, r, z);\ r \in I_j;\ j = \overline{1, n+1},$$

with initial conditions

$$u_{jm,ik}(t,r,z)\Big|_{t=0} = g_{jm,ik}(r,z); \quad r \in I_j; \quad j = \overline{1,n+1}, \tag{14}$$

boundary conditions

$$\left.\frac{\partial^s u_{jm,ik}}{\partial z^s}\right|_{z=-\infty} = 0; \quad \left.\frac{\partial^s u_{jm,ik}}{\partial z^s}\right|_{z=+\infty} = 0; \quad s = 0,1; \quad j = \overline{1,n+1}, \tag{15}$$

$$\begin{gathered}\left.\left(\alpha_{11}^0 \frac{\partial}{\partial r} + \beta_{11}^0\right) u_{1m,ik}\right|_{r=R_0} = g_{0m,ik}(t,z); \\ \left.\left(\alpha_{22}^{n+1} \frac{\partial}{\partial r} + \beta_{22}^{n+1}\right) u_{n+1m,ik}\right|_{r=R} = g_{m,ik}(t,z)\end{gathered} \tag{16}$$

and conjugate conditions

$$\begin{gathered}\left.\left[\left(\alpha_{j1}^p \frac{\partial}{\partial r} + \beta_{j1}^p\right) u_{pm,ik} - \left(\alpha_{j2}^p \frac{\partial}{\partial r} + \beta_{j2}^p\right) u_{p+1,m,ik}\right]\right|_{r=R_p} = 0; \\ j = 1,2; \quad p = \overline{1,n},\end{gathered} \tag{17}$$

here $\nu_{jm,ik} = a_{rj}^{-1} a_{\varphi j} \beta_{m,ik}$;

$$G_{jm,ik}(t,r,z) = f_{m,ik}(t,r,z) + a_{\varphi j}^2 r^{-2} \Phi_{m,ik}(t,r,z).$$

Let's apply to the two-dimensional initial-boundary value problem of conjugation (13)-(17) integral Fourier transform on the Cartesian axis $(-\infty;+\infty)$ relative to the variable z [20]:

$$F\left[g(z)\right] = \int_{-\infty}^{+\infty} g(z)\exp(-i\sigma z)dz \equiv \tilde{g}(\sigma), \quad i = \sqrt{-1}, \tag{18}$$

$$F^{-1}[\tilde{g}(\sigma)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{g}(\sigma)\exp(i\sigma z)d\sigma \equiv g(z), \tag{19}$$

$$F\left[\frac{d^2 g}{dz^2}\right] = -\sigma^2 F[g(z)] \equiv -\sigma^2 \tilde{g}(\sigma). \tag{20}$$

The integral operator F due to the formula (18) as a result of identity (20) boundary value problem (13)-(17) puts in accordance the task of constructing classical solution of one-dimensional differential equations of B — parabolic type of the 2nd order which is limited in the set $D'' = \left\{(t,r) : t > 0; r \in I_n^+\right\}$

$$\begin{gathered}\frac{\partial \tilde{u}_{jm,ik}}{\partial t} - a_{rj}^2 B_{\nu_{jm,ik}}[\tilde{u}_{jm,ik}] + \left(a_{zj}^2\sigma^2 + \chi_j^2\right)\tilde{u}_{jm,ik} = \\ = \tilde{G}_{jm,ik}(t,r,\sigma); \; r \in I_j; \; j = \overline{1,n+1}\end{gathered} \tag{21}$$

with initial conditions

$$\tilde{u}_{jm,ik}(t,r,\sigma)\Big|_{t=0} = \tilde{g}_{jm,ik}(r,\sigma); \;\; r \in I_j; \quad j = \overline{1,n+1}, \tag{22}$$

boundary conditions

$$\begin{aligned} &\left(\alpha_{11}^{0}\frac{\partial}{\partial r} + \beta_{11}^{0}\right)\tilde{u}_{1m,ik}\Bigg|_{r=R_0} = \tilde{g}_{0m,ik}(r,\sigma); \\ &\left(\alpha_{22}^{n+1}\frac{\partial}{\partial r} + \beta_{22}^{n+1}\right)\tilde{u}_{n+1,m,ik}\Bigg|_{r=R} = \tilde{g}_{m,ik}(r,\sigma) \end{aligned} \tag{23}$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^{p}\frac{\partial}{\partial r} + \beta_{j1}^{p}\right)\tilde{u}_{pm,ik} - \left(\alpha_{j2}^{p}\frac{\partial}{\partial r} + \beta_{j2}^{p}\right)\tilde{u}_{p+1,m,ik}\right]\Bigg|_{r=R_0} = 0; \qquad j = 1,2; \;\; p = \overline{1,n}, \tag{24}$$

here $B_{\nu_{jm,ik}} = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{\nu_{jm,ik}^2}{r^2}$ — classical Bessel differential operator.

To the one-dimensional initial-boundary problem of conjugation (21)-(24) let's apply finite hybrid integral Hankel transform of 2nd kind relative to the radial variable r in piecewise homogeneous segment I_n^+ of n conjugation points [12]:

$$M_{sn}\left[f(r)\right] = \int_{R_0}^{R} f(r)V(r,\lambda_s)\sigma(r)rdr \equiv \tilde{f}(\lambda_s), \tag{25}$$

$$M_{sn}^{-1}\left[\tilde{f}(\lambda_s)\right] = \sum_{s=1}^{\infty}\tilde{f}(\lambda_s)\frac{V(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2} \equiv f(r), \tag{26}$$

$$\begin{aligned} M_{sn}\left[B_{(m,ik)}[f(r)]\right] = &-\lambda_s^2\tilde{f}(\lambda_s) - \sum_{k=1}^{n+1}\gamma_k^2\int_{R_{k-1}}^{R_k} f(r)V_k(r,\lambda_s)\sigma_k rdr - \\ &-a_1^2R_0\sigma_1\left(\alpha_{11}^{0}\right)^{-1}V_1\left(R_0,\lambda_s\right)\left(\alpha_{11}^{0}\frac{df}{dr} + \beta_{11}^{0}f\right)\Bigg|_{r=R_0} + \\ &+a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\left(\alpha_{22}^{n+1}\frac{df}{dr} + \beta_{22}^{n+1}f\right)\Bigg|_{r=R}. \end{aligned} \tag{27}$$

Spectral function $V(r,\lambda_s)$, weight function $\sigma(r)$ and hybrid Bessel differential operator $B_{(m,ik)} = \sum_{j=1}^{n} a_j^2\theta(r-R_{j-1})\theta(R_j - r)B_{\nu_{jm,ik}}$, written in [12], take part in formulas (25)-(27) ($\theta(x)$ is the Heaviside step function).

Let's write the differential equations (21) and the initial conditions (22) in matrix form

$$\begin{bmatrix} \left(\frac{\partial}{\partial t} - a_1^2 B_{\nu_{1m,ik}} + q_1^2(\sigma)\right)\tilde{u}_{1m,ik}(t,r,\sigma) \\ \left(\frac{\partial}{\partial t} - a_2^2 B_{\nu_{2m,ik}} + q_2^2(\sigma)\right)\tilde{u}_{2m,ik}(t,r,\sigma) \\ \cdots\cdots\cdots\cdots \\ \left(\frac{\partial}{\partial t} - a_{n+1}^2 B_{\nu_{n+1,m,ik}} + q_{n+1}^2(\sigma)\right)\tilde{u}_{n+1,m,ik}(t,r,\sigma) \end{bmatrix} = \begin{bmatrix} \tilde{G}_{1m,ik}(t,r,\sigma) \\ \tilde{G}_{2m,ik}(t,r,\sigma) \\ \cdots\cdots \\ \tilde{G}_{n+1,m,ik}(t,r,\sigma) \end{bmatrix}, \quad (28)$$

$$\left.\begin{bmatrix} \tilde{u}_{1m,ik}(t,r,\sigma) \\ \tilde{u}_{2m,ik}(t,r,\sigma) \\ \cdots\cdots \\ \tilde{u}_{n+1,m,ik}(t,r,\sigma) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \tilde{g}_{1m,ik}(r,\sigma) \\ \tilde{g}_{2m,ik}(r,\sigma) \\ \cdots\cdots \\ \tilde{g}_{n+1,m,ik}(r,\sigma) \end{bmatrix}, \quad (29)$$

here $q_j^2(\sigma) = a_{zj}^2\sigma^2 + \chi_j^2; \quad j = \overline{1, n+1}.$

Let's represente the integral operator H_{sn} which operates due to the formula (25) as an operator matrix-row:

$$M_{sn}[\cdots] = \left[\int_{R_0}^{R_1} \cdots V_1(r,\lambda_s)\sigma_1 r dr \quad \int_{R_1}^{R_2} \cdots V_2(r,\lambda_s)\sigma_2 r dr \quad \cdots \quad \int_{R_{n-1}}^{R_n} \cdots V_n(r,\lambda_s)\sigma_n r dr \quad \int_{R_n}^{R} \cdots V_{n+1}(r,\lambda_s)\sigma_{n+1} r dr \right]. \quad (30)$$

Let's apply the operator matrix-row (30) to the problem (28), (29) according to the matrix multiplication rule. As a result of the identity (27), we get a Cauchy problem for ordinary differential equations of the 1st order

$$\sum_{j=1}^{n+1}\left(\frac{d}{dt} + \lambda_s^2 + \gamma_j^2 + q_j^2(\sigma)\right)\tilde{\tilde{u}}_{jm,ik}(t,\lambda_s,\sigma) =$$

$$= \sum_{j=1}^{n+1}\tilde{\tilde{G}}_{jm,ik}(t,\lambda_s,\sigma) - a_1^2 R_0\sigma_1\left(\alpha_{11}^0\right)^{-1} V_1(R_0,\lambda_s)\tilde{g}_{0m,ik}(t,\sigma) + \quad (31)$$

$$+ a_{n+1}^2 R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1} V_{n+1}(R,\lambda_s)\tilde{g}_{m,ik}(t,\sigma),$$

$$\left.\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jm,ik}(t,\lambda_s,\sigma)\right|_{t=0} = \sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jm,ik}^{1}(\lambda_s,\sigma), \quad (32)$$

where $\tilde{\tilde{u}}_{jm,ik}(t,\lambda_s,\sigma) = \int\limits_{R_{j-1}}^{R_j} \tilde{u}_{jm,ik}(t,r,\sigma)V_j(r,\lambda_s)\sigma_j r dr; \quad j=\overline{1,n+1},$

$$\tilde{\tilde{G}}_{jm,ik}(t,\lambda_s,\sigma) = \int\limits_{R_{j-1}}^{R_j} \tilde{G}_{jm,ik}(t,r,\sigma)V_j(r,\lambda_s)\sigma_j r dr; \quad j=\overline{1,n+1},$$

$$\tilde{\tilde{g}}_{jm,ik}(\lambda_s,\sigma) = \int\limits_{R_{j-1}}^{R_j} \tilde{g}_{jm,ik}(r,\sigma)V_j(r,\lambda_s)\sigma_j r dr, \quad j=\overline{1,n+1}.$$

Let's suppose that $\max\left\{q_1^2(\sigma), q_2^2(\sigma),...,q_{n+1}^2(\sigma)\right\} = q_1^2(\sigma)$ and put everywhere $\gamma_j^2 = q_1^2(\sigma) - q_j^2(\sigma); \quad j=\overline{1,n+1}$. Cauchy problem (31), (32) takes the form

$$\frac{d\tilde{\tilde{u}}_{m,ik}}{dt} + \Delta^2(\lambda_s,\sigma)\tilde{\tilde{u}}_{m,ik} = \tilde{\tilde{G}}_{m,ik}(t,\lambda_s,\sigma) - \\ -a_1^2 R_0\sigma_1\left(\alpha_{11}^0\right)^{-1} V_1(R_0,\lambda_s)\tilde{g}_{0m,ik}(t,\sigma) + \\ +a_{n+1}^2 R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1} V_{n+1}(R,\lambda_s)\tilde{g}_{m,ik}(t,\sigma), \tag{33}$$

$$\tilde{\tilde{u}}_{m,ik}(t,\lambda_s,\sigma)\Big|_{t=0} = \tilde{\tilde{g}}_{m,ik}(\lambda_s,\sigma), \tag{34}$$

here $\tilde{\tilde{u}}_{m,ik}(t,\lambda_s,\sigma) = \sum\limits_{j=1}^{n+1}\tilde{\tilde{u}}_{jm,ik}(t,\lambda_s,\sigma);$

$$\tilde{\tilde{G}}_{m,ik}(\lambda_s,\sigma) = \sum_{j=1}^{n+1}\tilde{\tilde{G}}_{jm,ik}(\lambda_s,\sigma); \quad \tilde{\tilde{g}}_{m,ik}(\lambda,\sigma) = \sum_{j=1}^{n+1}\tilde{\tilde{g}}^1_{jm,ik}(\lambda,\sigma);$$

$$\Delta^2(\lambda_s,\sigma) = \lambda_s^2 + a_{z1}^2\sigma^2 + \chi_1^2.$$

We can directly check that the only solution of the Cauchy problem (33), (34) is the function

$$\tilde{\tilde{u}}_{m,ik}(t,\lambda_s,\sigma) = N(t,\lambda_s,\sigma)\tilde{\tilde{g}}_{m,ik}(\lambda_s,\sigma) + \int\limits_0^t N(t-\tau,\lambda_s,\sigma)\times \\ \times\Bigg[\tilde{\tilde{G}}_{m,ik}(t,\lambda_s,\sigma) - a_1^2 R_0\sigma_1\left(\alpha_{11}^0\right)^{-1} V_1(R_0,\lambda_s)\tilde{g}_{0m,ik}(t,\sigma) + \\ +a_{n+1}^2 R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1} V_{n+1}(R,\lambda_s)\tilde{g}_{m,ik}(t,\sigma)\Bigg]d\tau, \tag{35}$$

here $N(t,\lambda_s,\sigma) = \exp(-\Delta^2(\lambda_s,\sigma)t)$. is solving function (Cauchy function).

The superposition of operators M_{sn} and M_{sn}^{-1} is a single operator $(M_{sn} \circ M_{sn}^{-1} = M_{sn}^{-1} \circ M_{sn} = I)$. Let's represent the operator M_{sn}^{-1} as inverse to operator (30), as the operator matrix-column:

$$M_{sn}^{-1}[\cdots] = \begin{bmatrix} \sum_{s=1}^{\infty} \cdots \dfrac{V_1(r,\lambda_s)}{\|V(r,\lambda_s)\|^2} \\ \sum_{s=1}^{\infty} \cdots \dfrac{V_2(r,\lambda_s)}{\|V(r,\lambda_s)\|^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{s=1}^{\infty} \cdots \dfrac{V_{n+1}(r,\lambda_s)}{\|V(r,\lambda_s)\|^2} \end{bmatrix} \tag{36}$$

Let's apply operator matrix-column (36) to the matrix-element $\left[\tilde{\tilde{u}}_{m,ik}(t,\lambda_s,\sigma)\right]$, where the function $\tilde{\tilde{u}}_{m,ik}(t,\lambda_s,\sigma)$ is defined by formula (35) due to the matrices multiplication rule. As a result we get the only solution of one-dimensional initial boundary problem of conjugation (21)-(24):

$$\begin{aligned} \tilde{u}_{jm,ik}(t,r,\sigma) &= \sum_{s=1}^{\infty} N(t,\lambda_s,\sigma)\tilde{\tilde{g}}_{m,ik}(\lambda_s,\sigma)\frac{V_j(r,\lambda_s)}{\|V(r,\lambda_s)\|^2} + \\ &+\sum_{s=1}^{\infty}\int_0^t N(t-\tau,\lambda_s,\sigma)\tilde{\tilde{G}}_{m,ik}(\tau,\lambda_s,\sigma)d\tau\frac{V_j(r,\lambda_s)}{\|V(r,\lambda_s)\|^2} + \\ &+\left(-a_1^2R_0\sigma_1\left(\alpha_{11}^0\right)^{-1}\right)\sum_{s=1}^{\infty}\int_0^t N(t-\tau,\lambda_s,\sigma)V_1(R_0,\lambda_s)g_{0m,ik}(\tau,\sigma)d\tau \times \\ &\times\frac{V_j(r,\lambda_s)}{\|V(r,\lambda_s)\|^2} + a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}\sum_{s=1}^{\infty}\int_0^t N(t-\tau,\lambda_s,\sigma)V_{n+1}(R,\lambda_s)\times \\ &\times\tilde{g}_{m,ik}(\tau,\sigma)\frac{V_j(r,\lambda_s)}{\|V(r,\lambda_s)\|^2}; \quad j=\overline{1,n+1}. \end{aligned} \tag{37}$$

If to apply consistently inverse operators F^{-1} and $F_{m,ik}^{-1}$ to functions $\tilde{u}_{jm,ik}(t,r,\sigma)$, which are defined by formulas (37) and perform the some simple transformations, we get functions

$$\begin{aligned} u_{j,ik}(t,r,\varphi,z) &= \sum_{p=1}^{n+1}\int_0^t\int_{R_{p-1}}^{R_p}\int_0^{\varphi_0}\int_{-\infty}^{+\infty} E_{jp}^{ik}(t-\tau,r,\rho,\varphi,\alpha,z-\xi)f_p(\tau,\rho,\alpha,\xi)\times \\ &\times\sigma_p\rho d\xi d\alpha d\rho d\tau + \sum_{p=1}^{n+1}\int_{R_{p-1}}^{R_p}\int_0^{\varphi_0}\int_{-\infty}^{+\infty} E_{jp}^{ik}(t,r,\rho,\varphi,\alpha,z-\xi)g_p(\rho,\alpha,\xi)\times \end{aligned}$$

$$\times\sigma_p \rho d\xi d\alpha d\rho + \sum_{p=1}^{n+1} a_{\varphi p}^2 \int_0^t \int_{R_{p-1}}^{R_p} \int_{-\infty}^{+\infty} Q_{jp}^{ik}(t,\tau,r,\rho,\varphi,z,\xi)\sigma_p \rho^{-1} d\xi d\rho d\tau + \quad (38)$$

$$+\int_0^t \int_0^{\varphi_0} \int_{-\infty}^{+\infty} \Big[W_{jr,ik}^1(t-\tau,r,\varphi,\alpha,z-\xi) g_0(\tau,\alpha,\xi) +$$

$$+W_{jr,ik}^2(t-\tau,r,\varphi,\alpha,z-\xi) g(\tau,\alpha,\xi) \Big] d\xi d\alpha d\tau;\ \ j=\overline{1,n+1}.$$

Functions (38) define the only solutions of parabolic initial boundary problems of conjugation (1)-(4), (5), (9); (1)-(4), (6), (9); (1)-(4), (7), (9); (1)-(4), (8), (9) with appropriate values of ik (11, 12, 21, 22).

In formulas (38) there are components

$$E_{jp}^{ik}(t,r,\rho,\varphi,\alpha,z) = \frac{2}{\pi\varphi_0} \sum_{m=0}^{\infty} \varepsilon_m^{ik} K_{jp}^{m,ik}(t,r,\rho,z) U_{m,ik}(\varphi) U_{m,ik}(\alpha)$$

of matrix of influence (functions of influence), Green's functions

$$Q_{jp}^{ik}(t,\tau,r,\rho,\varphi,z,\xi) = \frac{2}{\pi\varphi_0} \sum_{m=0}^{\infty} \varepsilon_m^{ik} K_{jp}^{m,ik}(t-\tau,r,\rho,z-\xi) \Phi_{m,ik}(\tau,\rho,\xi) U_{m,ik}(\varphi),$$

components

$$W_{jr,ik}^1(t,r,\varphi,\alpha,z) = -a_1^2 R_0 \sigma_1 \left(\alpha_{11}^0\right)^{-1} E_{j1}(t,r,R_0,\varphi,\alpha,z)$$

of left radial Green's matrix (left radial Green's functions) and components

$$W_{jr,ik}^2(t,r,\varphi,\alpha,z) = a_{n+1}^2 R \sigma_{n+1} \left(\alpha_{22}^{n+1}\right)^{-1} E_{j,n+1}^{ik}(t,r,R,\varphi,\alpha,z)$$

of right radial Green's matrix (right radial Green's functions) of corresponding initial-boundary problems of conjugation, here

$$K_{jp}^{m,ik}(t,r,\rho,z) = \sum_{s=1}^{\infty} \int_0^{+\infty} N(t,\lambda_s,\sigma) \cos(\sigma z) d\sigma \frac{V_j(r,\lambda_s) V_k(\rho,\lambda_s)}{\|V(r,\lambda_s)\|^2}.$$

Let's analyze formulas (38) depending on the type of boundary conditions on the wedge boundaries of an unbounded piecewise homogeneous wedge-shaped hollow cylinder. Let's consider, for example, the case of boundary conditions (6). In this case, Green's functions

$$Q_{jp}^{12}(t,\tau,r,\rho,\varphi,z,\xi) = \frac{2}{\varphi_0^2} \sum_{m=1}^{\infty} K_{jp}^{m,12}(t-\tau,r,\rho,z-\xi) \times$$

$$\times\left[\frac{\pi(2m+1)}{2\varphi_0} g_{2p}(\tau,\rho,\xi) + (-1)^{m+1} w_{2p}(\tau,\rho,\xi) \right] \sin\frac{\pi(2m+1)\varphi}{2\varphi_0}.$$

Let's determine the tangential Green's functions generated by boundary conditions (6) by the formulas:

$$W_{jp,1}^{12}(t,\tau,r,\rho,\varphi,z,\xi)=\frac{1}{\varphi_0^2}\sum_{m=1}^{\infty}(2m+1)K_{jp}^{m,12}(t-\tau,r,\rho,z-\xi)\sin\frac{\pi(2m+1)\varphi}{2\varphi_0},$$

$$W_{jp,2}^{12}(t,\tau,r,\rho,\varphi,z,\xi)=\frac{2}{\pi\varphi_0}\sum_{m=1}^{\infty}(-1)^{m+1}K_{jp}^{m,12}(t-\tau,r,\rho,z-\xi)\sin\frac{\pi(2m+1)\varphi}{2\varphi_0}.$$

Then the solution of the problem of conjugation (1)-(4), (6), (9) we can write in the form

$$u_{j,12}(t,r,\varphi,z)=\sum_{p=1}^{n+1}\int_0^t\int_{R_{p-1}}^{R_p}\int_0^{\varphi_0}\int_{-\infty}^{+\infty}E_{jp}^{12}(t-\tau,r,\rho,\varphi,\alpha,z-\xi)f_p(\tau,\rho,\alpha,\xi)\times$$

$$\times\sigma_\rho\rho d\xi d\alpha d\rho d\tau+\sum_{p=1}^{n+1}\int_{R_{p-1}}^{R_p}\int_0^{\varphi_0}\int_{-\infty}^{+\infty}E_{jp}^{12}(t,r,\rho,\varphi,\alpha,z-\xi)g_p(\rho,\alpha,\xi)\times$$

$$\times\sigma_\rho\rho d\xi d\alpha d\rho+\sum_{p=1}^{n+1}a_{\varphi p}^2\int_0^t\int_{R_{p-1}}^{R_p}\int_{-\infty}^{+\infty}[W_{jp,1}^{12}(t,\tau,r,\rho,\varphi,z,\xi)g_{2p}(\tau,\rho,\xi)+$$

$$+W_{jp,2}^{12}(t,\tau,r,\rho,\varphi,z,\xi)w_{2p}(\tau,\rho,\xi)]\sigma_p\rho^{-1}d\xi d\rho d\tau+$$

$$+\int_0^t\int_0^{\varphi_0}\int_{-\infty}^{+\infty}\Big[W_{jr,12}^{1}(t-\tau,r,\varphi,\alpha,z-\xi)g_0(\tau,\alpha,\xi)+$$

$$+W_{jr,12}^{2}(t-\tau,r,\varphi,\alpha,z-\xi)g(\tau,\alpha,\xi)\Big]d\xi d\alpha d\tau;\quad j=\overline{1,n+1}.$$

Using a properties of functions of influence $E_{jp}^{12}(t,r,\rho,\varphi,\alpha,z)$ and Green's functions $W_{jp,s}^{12}(t,\tau,r,\rho,\varphi,z,\xi)$, $(s=1,2)$, $W_{jr,12}^{k}(t,r,\varphi,\alpha,z)$, $(k=1,2)$ we can verify that functions $u_{j,12}(t,r,\varphi,z)$ which are defined by formulas (39), satisfy the equation (1), the initial conditions (2), the boundary conditions (3), (4), (6) and conjugate conditions (9) in the sense of theory of generalized functions [23].

The uniqueness of the solution (39) follows from its structure (integral image) and from uniqueness of the main solutions (functions of influence and Green's functions) of parabolic initial-boundary value problem of conjugation (1)-(4), (6), (9).

By methods from [1, 23] can be proved that under appropriate conditions on the initial data, formulas (39) define a limited classical solution of the concidered problem (1)-(4), (6), (9).

We get the following theorem as the summary of the above results.

Theorem. If functions $f_j(t,r,\varphi,z)$, $g_j(r,\varphi,z)$, $g_{2j}(t,r,z)$, $w_{2j}(t,r,z)$, $(j=\overline{1,n+1})$ satisfy conditions:

1) are continuously differentiated by variable t and continuously differentiated twice by the geometric variables;
2) have a limited variation for the geometric variables;
3) are absolutely summable with the variable z in $(-\infty;+\infty)$;
4) conjugate conditions are true and functions $g_0(t,\varphi,z)$, $g(t,\varphi,z)$ also satisfy the conditions 1)-3), then the parabolic initial-boundary value problem of conjugation (1)-(4), (6), (9) has a single bounded classical solution, which is determined by formulas (39).

Cases of boundary conditions (5), (7) or (8) on the wedge boundaries $\varphi = 0$, $\varphi = \varphi_0$ we can analyze similarly.

Remark 1. In the case of $a_{rj} = a_{\varphi j} = a_{zj} \equiv a_j > 0$ formulas (38) define the structures of the solutions of considered problems in an isotropic unlimited piecewise homogeneous wedge-shaped hollow cylinder.

Remark 2. The case of changing φ within $\varphi_1 < \varphi < \varphi_2$ is reduced to the considered replacement $\varphi' = \varphi - \varphi_1$ $(\varphi_0 \equiv \varphi_2 - \varphi_1)$.

Remark 3. Parameters α_{11}^0, β_{11}^0 α_{22}^{n+1}, β_{22}^{n+1} allow to allocate from formulas (38) the solutions of initial boundary value problems of conjugation in the case of boundary conditions of the 1st kind, 2nd kind and 3rd kind and their possible combinations on the radial surfaces $r = R_0$, $r = R$.

Remark 4. Analysis of the solution (38) is done directly from the general structures according to the analytical expression of functions $f_j(t,r,\varphi,z)$, $g_j(r,\varphi,z)$, $g_{kj}(t,r,z)$, $w_{kj}(t,r,z)$, $j = \overline{1,n+1}$, $k = \overline{1,4}$, $g_0(t,\varphi,z)$, $g(t,\varphi,z)$.

Conclusions. By means of method of classical integral and hybrid integral transforms and with the method of principal solutions (influence functions and Green's functions) exact analytical solutions of parabolic boundary-value problems in unlimited piecewise homogeneous wedge-shaped hollow cylinder are obtained at first time. The obtained integrated images of solutions are of algorithmic character, continuously depend on the parameters and data of the problem and can be used both in further theoretical research and in the practice of engineering calculations of mathematical models of evolutionary processes in piecewise homogeneous environments.

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ПАРАБОЛІЧНІ КРАЙОВІ ЗАДАЧІ В НЕОБМЕЖЕНОМУ КУСКОВО-ОДНОРІДНОМУ КЛИНОВИДНОМУ ПОРОЖНИСТОМУ ЦИЛІНДРІ

У пропонованій статті методом класичних інтегральних і гібридних інтегральних перетворень у поєднанні з методом головних розв'язків (матриць впливу та матриць Гріна) вперше побудовано єдині точні аналітичні розв'язки параболічних крайових задач математичної фізики в необмеженому за змінною z кусково-однорідному за радіальною змінною r клиновидному за кутовою змінною φ порожнистому циліндрі.

Розглянуто випадки задання на гранях клина крайових умов Діріхле і Неймана та їх можливих комбінацій (Діріхле — Неймана, Неймана — Діріхле).

Для побудови класичних розв'язків досліджуваних початково-крайових задач застосовано скінченне інтегральне перетворення Фур'є щодо кутової змінної, інтегральне перетворення Фур'є на декартовій осі щодо аплікатної змінної та гібридне інтегральне перетворення типу Ганкеля 2-го роду на сегменті полярної осі з n точками спряження щодо радіальної змінної.

Послідовне застосування інтегральних перетворень за геометричними змінними дозволяє звести тривимірні початково-крайові задачі спряження до задачі Коші для звичайного лінійного неоднорідного диференціального рівняння 1-го порядку, єдиний розв'язок якої виписано в замкнутому вигляді.

Застосування обернених інтегральних перетворень відновлює в явному вигляді розв'язки розглянутих задач через їх інтегральне зображення.

Ключові слова: *параболічне рівняння, початкові та крайові умови, умови спряження, інтегральні перетворення, гібридні інтегральні перетворення, головні розв'язки.*

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