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It has been established the sufficient conditions for the convergence of the continuous stochastic optimization procedure of Kiefer-Wolfowitz in the diffusion approximation scheme with Markov switching. The convergence of the proposed procedure has been proved by using the method of the small parameter and the solution of the singular perturbation problem for the Markov process generator.

Key words: *stochastic optimization, the markov process, solution of the singular perturbation problem.*

Отримано 16.10.2010

УДК 517.9

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ON TIME ASYMPTOTIC OF THE SOLUTIONS OF TRANSPORT EVOLUTION EQUATION

Authors consider in the space $L^2(D), D = R\times[-1,1]$ the transport operator

$$Lf = -i\mu\frac{\partial f}{\partial x} + a(x)b_1(\mu)\int_{-1}^{1} b(\mu')f\left(x,\mu'\right)d\mu'.$$

To obtain the representation of the solution of the equation $\dot{u} = iLu, u\big|_{t=0} = u(0)$ authors introduce some integral (like known expression the semigroup by the resolvent) and prove directly that this integral is corresponding semigroup. To simplify the calculus the authors reduce the operator L to some Friedrichs' model using Fourier transformation.

Key words: *spectrum, transport operator, Friedrichs' model, semigroup.*

Spectral theory for various types of transport operators is the subject of many works (see for example [1; 2]). In the work [3] the authors begin to use Friedrichs' model to study the spectrum of some transport operator. Analogic result was obtained in the works [4; 5].

In the present article for such operators corresponding evolution is considered. We consider in the space $L^2(D)$, $D = R\times[-1,1]$ the transport operator

$$Lf = -i\mu\frac{\partial f}{\partial x} + a(x)b_1(\mu)\int_{-1}^{1} b(\mu')f(x,\mu')d\mu' \tag{1}$$

with maximum domain of definition $D(L)$ under the following conditions: there exist constants $M > 0, \varepsilon > 0$ such that

$$\left|a(x)\right| < Me^{-\varepsilon|x|},\ x\in R \tag{2}$$

and the functions $b(\mu), b_1(\mu)$ admit analytic prolongation from interval (-1,1) into the circle $\left|z\right| < 1+\varepsilon$. Note thaat in [4; 5] we have $b_1(\mu)\equiv 1$ and in [3] $b_0(\mu)\equiv b(\mu)\equiv 1$.

We will study the following transport evolution equation

$$\begin{cases} \dot{u} = iLu, & t>0 \\ u\big|_{t=0} = u(0), & u(0)\in D(L) \end{cases} \tag{3}$$

and obtain principal term of asymptotic behaviour of the solutions of this equation corresponding to eigen-values of L. We suppose that the operator L has not spectral singularities.

1. Friedrichs' model of transport operator

We want to transform the operator L in Friedrichs' model. We begin by the notations. Let H be Hilbert space of the functions $\varphi(s,\mu), (s,\mu)\in D, D = R\times[-1,1]$ with the norm

$$\left\|\varphi\right\|_H^2 = \int_R\int_{-1}^{1}\left|\varphi(s,\mu)\right|^2\frac{1}{|\mu|}dsd\mu.$$

We introduce the operator $F_0 : L^2(D)\to H$, where

$$\left(F_0u\right)(\tau,\mu) = u\left(\frac{\tau}{\mu},\mu\right), u\in L^2(D), \tau\in R \tag{4}$$

and the operator $Z : L^2(R)\to H$, where

$$(Zc)(\tau,\mu) = c\left(\frac{\tau}{\mu}\right), c\in L^2(R). \tag{5}$$

It is not difficult to verify that $\left\|F_0u\right\|_H = \left\|u\right\|_{L^2(D)}$, that F_0 is unitary operator and that the operator Z is bounded, namely $\left\|z\right\|\le\sqrt{2}$. The Fourier transformation is denoted by

$$(Ff)(s) = \frac{1}{\sqrt{2\pi}} \int_R e^{-is\tau} f(\tau) d\tau,\ s \in R$$

in the space $L^2(R)$ and in the space $L^2(D)$ too.

Now we apply to the equality (1) the Fourier transformation with respect to variable x, then

$$(FLf)(\tau,\mu) = \tau\mu u(\tau,\mu) + \frac{b_1(\mu)}{\sqrt{2\pi}} \int_R \left(\int_{-1}^{1} a(x) b(\mu') f(x,\mu') d\mu' \right) e^{-i\tau x} dx,$$

where $u = Ff$.

Now we apply the operator F_0 (see(4)), simply it is the substitution $\tau = \frac{s}{\mu}$, then

$$(F_0 FLf)(s,\mu) = su\left(\frac{s}{\mu}, \mu \right) + \frac{b_1(\mu)}{\sqrt{2\pi}} \int_R \left(\int_{-1}^{1} a(x) b(\mu') f(x,\mu') d\mu' \right) e^{-ix\frac{s}{\mu}} dx.$$

We denote $\varphi(s,\mu) = u\left(\frac{s}{\mu}, \mu \right)$ or $\varphi = F_0 u = F_0 Ff$. Let $\frac{s}{\mu} = \tau$, then $u(\tau,\mu) = \varphi(\tau\mu,\mu) = (F_0^{-1}\varphi)(\tau,\mu)$. That's why

$$f(x,\mu) = (F^{-1}F_0^{-1}\varphi)(x,\mu) = \frac{1}{\sqrt{2\pi}} \int_R \varphi(\tau\mu,\mu) e^{i\tau x} d\tau.$$

The change of variable $s' = \mu\tau$ (cases $\mu > 0$ and $\mu < 0$) gives

$$f(x,\mu') = \frac{1}{\sqrt{2\pi}} \int_R \varphi(s',\mu') e^{i\frac{s'}{\mu'}x} \frac{ds'}{|\mu'|}.$$

Finally, we obtain

$$\left(F_0 FLF^{-1}F_0^{-1}\varphi \right)(s,\mu) = s\varphi(s,\mu) + V\varphi(s,\mu), \tag{6}$$

where

$$V\varphi(s,\mu) = \frac{b_1(\mu)}{2\pi} \int_R \int_{-1}^{1} a(x) b(\mu') \left(\int_R \varphi(s',\mu') e^{i\frac{s'}{\mu'}} ds' \right) \frac{d\mu'}{|\mu'|} e^{-ix\frac{s}{\mu}} dx. \tag{7}$$

We choose some factorization for the function $a(x)$ such that

$$a(x) = \overline{a_1(x)} a_2(x),\ |a_1(x)| = |a_2(x)|. \tag{8}$$

Let $G = L^2(R)$ then $V = A^* B$ (see (7)), where the operators $A, B : H \to G$ are given by the expressions

$$\begin{cases} A^{*}c(s,\mu)=\dfrac{1}{\sqrt{2\pi}}b_1(\mu)\displaystyle\int_R \overline{a_1(x)}c(x)e^{-ix\frac{s}{\mu}}dx, \\ B\varphi(x)=\dfrac{1}{\sqrt{2\pi}}a_2(x)\displaystyle\int_R\int_{-1}^{1} b(\mu')\varphi(s',\mu')e^{i\frac{s'}{\mu'}}\dfrac{d\mu'}{|\mu'|}ds'. \end{cases} \tag{9}$$

Obviously, the operator $U=F_0F:L^2(D)\to H$ (see(4)) is unitary. So, the following theorem is proved.

1.1. Theorem. Let $L:L^2(D)\to L^2(D)$ be the operator with maximal domain of definition, given by the expression (1). Then

$$ULU^{-1}=T:H\to H,$$

where

$$T=S+V, V=A^{*}B,$$

$(S\varphi)(\tau,\mu)\equiv\tau\varphi(\tau,\mu),\ \tau\in R,\ \mu\in(-1,1)$ and the operators A,B act from H into $G=L^2(R)$ (see (9)).

The integral operator in the right side of (1) is bounded in the space $L^2(D)$. So, the operator V is bounded in space H. However we need following Lemma.

1.2. Lemma. The operators $A,B:H\to G,$ namely

$$\begin{cases} A\varphi(x)=\dfrac{1}{\sqrt{2\pi}}a_1(x)\displaystyle\int_R\int_{-1}^{1} b_1(\mu)\varphi(s,\mu)e^{ix\frac{s}{\mu}}\dfrac{1}{|\mu|}d\mu ds \\ B\varphi(x)=\dfrac{1}{\sqrt{2\pi}}a_2(x)\displaystyle\int_R\int_{-1}^{1} b(\mu)\varphi(s,\mu)e^{ix\frac{s}{\mu}}\dfrac{1}{|\mu|}d\mu ds \end{cases} \tag{10}$$

are bounded.

1.3. Proposition. If $a(x)\ge 0,\ x\in R$ and $b_1(\mu)=b(\mu),\ \mu\in(-1,1)$, then Friedrichs' model $T=S+V$ is selfadjoint operator, $T^{*}=T$.

Proof. Really, as factorization $a(x)=\overline{a_1(x)}a_2(x)$ is arbitrary one can choose $a_1(x)=a_2(x)=\sqrt{a(x)}$, then according to (10) $A=B$, so $T^{*}=T$ in view of $S^{*}=S.$ Proposition is proved.

2. Spectrum of Friedrichs' model

We will consider the resolvent of the operator $T=S+A^{*}B.$ The equation $(T-\zeta)\varphi=\psi,\psi\in H,\zeta\notin R$ takes form $(S-\zeta)\varphi+A^{*}B\varphi=\psi.$

Denote $S_\zeta = (S-\zeta)^{-1}, T_\zeta = (T-\zeta)^{-1}$. As $\zeta \notin R$ then the operator S_ζ exists, is bounded and the equation takes form

$$\varphi + S_\zeta A^* B\varphi = S_\zeta \psi. \qquad (11)$$

Applying the operator B, we obtain $\left(1+BS_\zeta A^*\right)B\varphi = BS_\zeta\psi$. Let

$$K(\zeta) = 1 + BS_\zeta A^*, \zeta \notin R, \qquad (12)$$

then for the expression $B\varphi$ in (11) we obtain $B\varphi = K(\zeta)^{-1}BS_\zeta\psi$. Taking into account the boundness of the operators A and B, we have the following proposition.

2.1. Proposition. If the operator $K(\zeta), \zeta \notin R$ has bounded inverse operator $K(\zeta)^{-1}$, then the value ζ belongs to resolvent set of the operator T and

$$T_\zeta = S_\zeta - S_\zeta A^* K(\zeta)^{-1} BS_\zeta. \qquad (13)$$

Substituing the expressions (9) for the operators A and B into (12) we obtain immediately (after the change of variables $\tau = \frac{s'}{\mu'}$) the following Lemma.

2.2. Lemma. The operator $K(\zeta)$ (see (12)) admits the representation

$$((K(\zeta)-1)c)(x) = \int_R k(x,y,\zeta)c(y)dy, \zeta \notin R, \qquad (14)$$

where $k(x,y,\zeta) = \frac{1}{2\pi}a_2(x)\overline{a_1(y)}I(x-y,\zeta)$ and

$$I(u,\zeta) = \int_R l(\tau,\zeta)e^{iu\tau}d\tau,\ l(\tau,\zeta) = \int_{-1}^{1}\frac{b(\mu')b_1(\mu')}{\tau\mu'-\zeta}d\mu'. \qquad (15)$$

This Lemma coincides with corresponding Lemma of the work [4], where $b_1(\mu) \equiv 1$ and the function $l(\tau,\zeta)$ was (compare with (15))

$$l(\tau,\zeta) = \int_{-1}^{1}\frac{b(\mu')}{\tau\mu'-\zeta}d\mu'.$$

2.3. Theorem. Operator $K(\zeta)-1 : L^2(R) \to L^2(R), \zeta \notin R$ is compact and $\left\|K(\zeta)-1\right\| \to 0, \left|\zeta\right| \to \infty$ uniformly in the domain $\left|\operatorname{Im}\zeta\right| > 0$ for every $\nu > 0$.

In the work [4] it was shown that the proof of this Theorem practically coincides with the corresponding proof in the work [3].

2.4. Theorem. Operator $K(\zeta)-1$ admits analytic prolongation $K_{\pm}(\zeta)-1$ over semiaxes $(-\infty,0)$ and $(0,\infty)$ and $\|K_{\pm}(\zeta)-1\|\to 0$, $|\zeta|\to\infty$ uniformly in the domain $|\operatorname{Im}\zeta|<\varepsilon$ for every $\varepsilon_1<\dfrac{\varepsilon}{2}$.

We have the same proof as the proof in the work [4], instead of the function $b(\mu)$ one uses the function $b_0(\mu)\equiv b(\mu)b_1(\mu)$. This proof is based on the representation

$$I(u,\zeta)=\int_0^\infty \frac{1}{t-\zeta} f_{-\omega}\left(t|u|\right)dt-\int_0^\infty \frac{1}{t+\zeta} f_{\omega}\left(t|u|\right)dt,\ \omega=\operatorname{sign} u,$$

where

$$f_\omega(\theta)=\int_\theta^\infty \frac{1}{y}\left[b_0\left(\frac{\theta}{y}\right)e^{-i\omega y}+b_0\left(-\frac{\theta}{y}\right)e^{i\omega y}\right]dy,\ \omega=\pm 1.$$

Using well known Theorem on holomorphic operator function (see [6], Ch. 1) and Theorem 2.4, we obtain that the point spectrum outside of R can have point of accumulation $\zeta=0$ only. The condition of finiteness of such spectrum we can give by analogy to the work [5]. Later we will suppose that point spectrum of the operator T is finite. At the end we indicate only that continuous spectrum and spectral singularities of the operator T belong to the axis R.

3. Construction of the semi-group $\exp(itT)$

It is known that Cauchy problem $\dot{u}=Mu,\ u\big|_{t=0}=u(0),$ where the operator M is such that half plane $\operatorname{Re}\zeta>\gamma>0$ belongs to its resolvent set admits under some condition following representation of the solution

$$u(t)=-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\zeta t}R(\zeta,M)u(0)d\zeta,\ \ R(\zeta,M)=(M-\zeta)^{-1}.$$

Let us consider the problem

$$\begin{cases}\dot{u}=iTu,\ t>0,\\ u\big|_{t=0}=u(0),\ u(0)\in D(T).\end{cases}\tag{16}$$

Denote $\zeta=\gamma+i\theta,\ -\infty<\theta<\infty,$ then $(iT-\zeta)^{-1}=-iT_{\theta-i\gamma}$.

Instead of difficult verification of some sufficient condition on the operator T, we propose directly to choose the solution under the form

$$u(t)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\gamma+i\theta)t}T_{\theta-i\gamma}u(0)d\theta.\tag{17}$$

We denote

$$h(t,\theta)=e^{(\gamma+i\theta)t}$$

and we denote the element $u(0)=\varphi(\tau,\mu)\in H$ simply by $u(0)=\varphi.$ So, we must prove that the operator

$$U(t)\varphi=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}h(t,\theta)T_{\theta-i\gamma}d\theta,\ t>0 \tag{18}$$

defines semi-group corresponding to the problem (16).

3.1. Theorem. If $\varphi\in D(T)$ then the integral (18) admits the representation

$$U(t)\varphi=\varphi-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h(t,\theta)}{\theta-i\gamma}T_{\theta-i\gamma}T\varphi\,d\theta,\ t>0, \tag{19}$$

where the integral converges with respect to the norm of H .

Proof. To prove this theorem it is sufficient to substitute in (17) such an expression:

$$\frac{1}{\theta-i\gamma}\left[(\theta-i\gamma)-T+T\right]=1.$$

3.2. Lemma. If $\varphi\in D(T)$ then

$$\int_{-\infty}^{\infty}\frac{h(t,\theta)}{\theta-i\gamma}T_{\theta-i\gamma}T\varphi\,d\theta=T\int_{-\infty}^{\infty}\frac{h(t,\theta)}{\theta-i\gamma}T_{\theta-i\gamma}\varphi\,d\theta. \tag{20}$$

Proof. As the operator T is closed it's sufficient to replace the integral by the corresponding integral sum. Denote $\frac{\Delta h}{\Delta t}(\theta)= \ =(h(t+\Delta t,\theta)- -h(t,\theta))/\Delta t$ .As $h(t,\theta)=\exp((\gamma+i\theta)t)$, then $\frac{\partial}{\partial t}h(t,\theta)=i(\theta-i\gamma)h(t,\theta).$

3.3. Lemma. If $\varphi\in H$, then

$$s-\lim_{\Delta t\to 0}\int_{-\infty}^{\infty}\frac{1}{\theta-i\gamma}\frac{\Delta h}{\Delta t}(\theta)T_{\theta-i\gamma}\varphi d\theta=i\int_{-\infty}^{\infty}h(t,\theta)T_{\theta-i\gamma}\varphi\,d\theta. \tag{21}$$

Note, that the equality (21) in view of definition (18) signifies that

$$s-\lim_{\Delta t\to 0}\int_{-\infty}^{\infty}\frac{1}{\theta-i\gamma}\cdot\frac{\Delta h}{\Delta t}(\theta)T_{\theta-i\gamma}\varphi\,d\theta-2\pi U(t)\varphi,\ \varphi\in D(T). \tag{22}$$

3.4. Theorem. Let $\varphi\in D(T),$ then

$$U'(t)\varphi=iTU(t)\varphi,\ t>0, \tag{23}$$

where $U'(t)$ signifies strong derivative.

Proof. Using the representation (19) and the equality (20) we obtain two relations:

$$\frac{\Delta U(t)}{\Delta t}\varphi = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\theta - i\gamma}\cdot\frac{\Delta h}{\Delta t}(\theta)T_{\theta - i\gamma}T\varphi\, d\theta \tag{24}$$

and

$$\frac{\Delta U(t)}{\Delta t}\varphi = -\frac{1}{2\pi i}T\int_{-\infty}^{\infty}\frac{1}{\theta - i\gamma}\cdot\frac{\Delta h}{\Delta t}(\theta)T_{\theta - i\gamma}\varphi\, d\theta. \tag{25}$$

In view of Lemma 3.3, there exists strong limit of the integral in (24) if $\Delta t \to 0,$ so there exists strong limit

$$U'(t)\varphi = s - \lim_{\Delta t \to 0}\frac{\Delta U(t)}{\Delta t}\varphi,\ \varphi \in D(T).$$

By analogy, there exist strong limit of the integral in (25), which is equal to $2\pi U(t)\varphi$ (see (22)). As the operator T is closed, we obtain the equality (23) from the equalities (24)—(25). Theorem is proved.

3.5. Theorem. Let $\varphi \in D(T),$ then

$$\lim_{t \to 0}\left\| U(t)\varphi - \varphi \right\| = 0. \tag{26}$$

Proof. Recall the representation (19)

$$U(t)\varphi = \varphi - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h(t,\theta)}{\theta - i\gamma}T_{\theta - i\gamma}T\varphi d\theta,\ \ h(t,\theta) = e^{(\gamma + i\theta)t}. \tag{27}$$

First we prove that

$$\int_{-\infty}^{\infty}\frac{1}{\theta - i\gamma}T_{\theta - i\gamma}\eta\, d\theta = \mathrm{O},\ \ \eta = T\varphi. \tag{28}$$

Note, that the value $\gamma > 0$ is such that the operator function $z \to T_{z - i\gamma}\varphi_0$, $z = \theta + i\nu$ is holomorphic in half plane $\mathrm{Im}\, z < -\gamma$.

Let $L_R = \left\{ z : z = Re^{i\alpha},\ \pi < \alpha < 2\pi \right\}$ be semicircle such that $R \to \infty$. We obtain (28) if we prove that

$$\lim_{R \to \infty}\int_{L_R}\frac{1}{z - i\gamma}T_{z - i\gamma}\eta\, dz = 0. \tag{29}$$

As $\left\| K(z - i\gamma)^{-1} \right\| \le M,\ M = \mathrm{const}$ and $\left\| S_{z - i\gamma} \right\| \le \dfrac{1}{\left| \mathrm{Im}\, z \right| + \gamma}$, where $\mathrm{Im}\ z < 0$, then we have

$$\left\|T_{z-i\gamma}\eta\right\| = \left\|S_{z-i\gamma}\eta - S_{z-i\gamma}A^{*}K(z-i\gamma)^{-1}BS_{z-i\gamma}\eta\right\| \le$$
$$\le M_0\left\|S_{z-i\gamma}\eta\right\| \le \frac{M_0}{\left|\operatorname{Im} z\right| + \gamma},\ \ M_0 = \text{const}. \tag{30}$$

As $\left|\operatorname{Im} z\right| = R\left|\sin\alpha\right|$, the estimate of the integral (29) reduces to the following estimate

$$\int_{L_R} \frac{1}{\left|z-i\gamma\right|}\cdot\frac{\left|dz\right|}{\left|\operatorname{Im} z\right|+\gamma} \le \left(\int_{R\left|\sin\alpha\right|>\sqrt{R}} + \int_{R\left|\sin\alpha\right|<\sqrt{R}}\right)\frac{d\alpha}{R\left|\sin\alpha\right|+\gamma} \le$$
$$\le \int_{\left|\sin\alpha\right|>\frac{1}{\sqrt{R}}} \frac{d\,\alpha}{\sqrt{R}+\gamma} + \int_{\left|\sin\alpha\right|<\frac{1}{\sqrt{R}}} \frac{d\,\alpha}{\gamma} = O\left(\frac{1}{\sqrt{R}}\right),\ R\to\infty.$$

So, the relations (28) and (29) are proved. Now, taking into account (27)—(28) to prove the relation (26) we must prove that

$$\lim_{t\to 0}\left\|\int_{-\infty}^{\infty}\frac{h(t,\theta)-1}{\theta-i\gamma}\,T_{\theta-i\gamma}\eta\,d\theta\right\| = 0. \tag{31}$$

As $\left\|T_{\theta-i\gamma}\eta\right\| \le M_0\left\|S_{\theta-i\gamma}\eta\right\|$ (see(30)), it is sufficient to estimate the following integral

$$I(t) \equiv \int_{-\infty}^{\infty}\left|\frac{e^{(\gamma+i\theta)t}-1}{\gamma+i\theta}\right|\left\|S_{\theta-i\gamma}\eta\right\|d\theta \le$$
$$\le \left(\int_{-\infty}^{\infty}\left|\frac{e^{(\gamma+i\theta)t}-1}{\gamma+i\theta}\right|^2 d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left\|S_{\theta-i\gamma}\eta\right\|^2 d\theta\right)^{\frac{1}{2}}.$$

The function $\theta \to \left|e^{(\gamma+i\theta)t}-1\right|/\left|\gamma+i\theta\right|$ is majorable by some function from $L^2(R)$ uniformly in every finite interval $0<t<t_1$. In addition, the integral

$$\int_{-\infty}^{\infty}\left\|S_{\theta-i\gamma}\eta\right\|^2 d\theta = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-1}^{1}\frac{\left|\eta(\tau,\mu)\right|^2}{\left|\tau-(\theta-i\gamma)\right|^2}\cdot\frac{1}{\left|\mu\right|}d\tau d\mu\right)d\theta \le$$
$$\le \frac{\pi}{\gamma}\int_{-\infty}^{\infty}\int_{-1}^{1}\left|\eta(\tau,\mu)\right|^2\cdot\frac{1}{\left|\mu\right|}d\tau d\mu = \frac{\pi}{\gamma}\left\|\eta\right\|^2 < \infty$$

converges. So, $I(t)\to 0, t\to 0$, what proves the relation (31). Theorem is proved.

3.6. Proposition. If

$$\begin{cases} \dot{u} = iTu, t > 0, \\ u\big|_{t=0} = 0, \end{cases}$$

then $u(t) \equiv 0, t > 0$.

As corrolary, we have that the operator $U(t), t > 0$ (see (18)) defines a semigroup.

Now we can formulate the main theorem of this section.

3.7. Theorem. The problem

$$\begin{cases} \dot{u} = iTu, t > 0, \\ u\big|_{t=0} = \varphi, \varphi \in D(T), \end{cases}$$

has unique solution $u(t) = U(t)\varphi$, given by the semigroup

$$U(t)\varphi = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\gamma+i\theta)t} T_{\theta-i\gamma}\varphi \, d\theta.$$

Proof. Results from (23), (26) and Proposition 3.6.

4. Asymptotic behavior of the semigroup

We study time asymptotic of the solution of evolution equation $\dot{f} = iLf$, $f\big|_{t=0} = f(0)$. If the operator $L : L^2(D) \to L^2(D)$ is self-adjoint then corresponding semi-group is unitary and this solution is bounded with respect to the norm of the space. We consider nonself-adjoint operator L. Our aim is to give simple description of increasing part of the solution when $t \to \infty$.

It is convinient to use Friedrichs` model T (T is unitary equivalent to L) and study the solution of the problem $\dot{u} = iTu, u\big|_{t=0} = u(0)$.

Suppose that the following conditions hold:

a) $b(0) = b_1(0) = 0$;
b) the operator T has not spectral singularities, set of eigen-values is finite.

We have need of the following known statement (see, for ex. [7]).

4.1. Lemma. If $g \in L^2(R)$ and $\tilde{g}(\zeta) = \frac{1}{2\pi i}\int_R \frac{g(s)}{s-\zeta} ds$, then

$$\int_R \left|\tilde{g}(\sigma + i\tau)\right|^2 d\sigma \le \|g\|^2_{L^2(R)}, \tau \ne 0. \tag{32}$$

4.2. Lemma. Vector-functions $AS_\zeta\varphi, BS_\zeta\varphi, \varphi \in H$ belong to Hardy space in half planes $\operatorname{Im}\zeta > 0$ and $\operatorname{Im}\zeta < 0$.

Proof. Recall that (see(9))

$$B\varphi(x) = \frac{1}{\sqrt{2\pi}} a_2(x) \int_R \int_{-1}^{1} b(\mu) e^{ix\frac{s}{\mu}} \varphi(s,\mu) \frac{d\mu}{|\mu|} ds.$$

So,

$$BS_{\zeta}\varphi(x) = a_2(x) \int_R \frac{1}{s-\zeta} \Phi(s,x) ds, \tag{33}$$

where

$$\Phi(s,x) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \varphi(s,\mu) \frac{b(\mu)}{|\mu|} e^{ix\frac{s}{\mu}} d\mu.$$

As $b(0) = 0$, then

$$\sqrt{2\pi}\,|\Phi(s,x)| \le \int_{-1}^{1} |\phi(s,\mu)| \frac{|b(\mu)|}{|\mu|} d\mu \le \left(\int_{-1}^{1} |\phi(s,\mu)|^2 \frac{d\mu}{|\mu|} \right)^{\frac{1}{2}} \left(\int_{-1}^{1} \frac{|b(\mu)|^2}{|\mu|} d\mu \right)^{\frac{1}{2}}$$

and

$$\|\Phi(\bullet,x)\|_{L^2(R)}^2 = \int_R |\Phi(s,x)|^2 ds \le M \|\varphi\|_H^2, \; x \in R.$$

Using (32), we have

$$\int_R \left| \int_R \frac{1}{s-(\sigma+i\tau)} \Phi(s,x) ds \right|^2 d\sigma \le 4\pi^2 \|\Phi(\bullet,x)\|_{L^2(R)}^2 \le 4M\pi^2 \|\varphi\|_H^2, \; x \in R. \tag{34}$$

The representation (33) gives

$$\|BS_{\sigma+i\tau}\varphi\|^2 = \int_R |a_2(x)|^2 \left| \int_R \frac{1}{s-(\sigma+i\tau)} \Phi(s,x) ds \right|^2 dx.$$

Integrating this equality with respect to σ and using the estimate (34), which does not depend on x, we obtain

$$\int_R \|BS_{\sigma+i\tau}\varphi\|^2 d\sigma \le 4M\pi^2 \|\varphi\|_H^2 \int_R |a_2(x)|^2 dx \le M_1 \|\varphi\|_H^2, \; \tau \ne 0. \tag{35}$$

The vector-function $AS_{\zeta}\varphi$ is considered by analogy.

Lemma is proved.

Let $h(\zeta), \zeta = \sigma + i\tau$ be function holomorphic in half planes $\tau > 0$ and $\tau < 0$.

4.3. Lemma. Suppose that

$$\int_R |h(\sigma+i\tau)| d\sigma \le M, \; M = \text{const}, \; \tau \ne 0, \tag{36}$$

then the integral $H(\tau) = \int_R h(\sigma+i\tau) d\sigma$ does not depend on τ in the intervals $\tau > 0$ and $\tau < 0$.

Now we come back to the solution $u(t,\tau,\mu)$ of evolution equation (see Theorem 3.7). We will write $u(t)$ instead of $u(t,\tau,\mu)$. As $T_\zeta = S_\zeta - S_\zeta A^* K(\zeta)^{-1} BS_\zeta$, then

$$u(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\gamma+i\theta)t} T_{\theta-i\gamma}\varphi \, d\theta = e^{iSt}\varphi + I(t,\gamma)\varphi, t > 0, \tag{37}$$

where

$$I(t,\gamma)\varphi = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\gamma+i\theta)t} S_{\theta-i\gamma} A^* K(\theta - i\gamma)^{-1} BS_{\theta-i\gamma}\varphi d\theta. \tag{38}$$

The eigen-values ζ_k of the operator T are the poles of the operator function $K(\zeta)^{-1}$. Recall that the operator $K(\zeta)-1$ is compact. Therefore, according to the theorem about holomorphic operator function (see [6]), the coefficients $Q_{k,-j}$ of principal part of Loran decomposition of $K(\zeta)^{-1}$, namely

$$K(\zeta)^{-1} = \sum_k Q_k(\zeta) + G(\zeta), \operatorname{Im} \zeta \neq 0, \tag{39}$$

$$Q_k(\zeta) = \frac{Q_{k,-m}}{(\zeta-\zeta_k)^m} + \ldots + \frac{Q_{k,-1}}{\zeta - \zeta_k}, \zeta \neq \zeta_k$$

are finite dimensional operators.

Operator function $G(\zeta)$ is bounded and gives in (38) the term $O(1), t \to \infty,$ (see (35)).

It remains to calculate directly (by residues) the terms of the expression (38), containing the operators $Q_{k,-j}$ then the poles $\zeta_k = \theta - i\gamma$ give the functions $\exp((\gamma + i\theta)t) = \exp(i\zeta_k t)$ and its derivates with respect to ζ give the factors t^p. Using unitary equivalence of semi groups $\exp(itL) = U^{-1}\exp(itT)U$, we obtain following main result concerrning initial Cauchy problem (3).

4.4. Theorem. Suppose that the operator L has finite set of eigen-values and has not spectral singularities. Then asymptotic behaviour of the solution of evolution equation (3) is

$$u(t) = \sum_{\operatorname{Im}\zeta_k < 0} e^{i\zeta_k t} \sum_p t^p (u(0), f_{k,p})_H \, g_{k,p} + O(1), t \to \infty,$$

where $O(1)$ signifies the value in the space $L^2(D)$ and $f_{k,p}, g_{k,p}$ denote some elements from $L^2(D)$.

5. Case of finite-dimensional kernel

Let us consider in the space $L^2(D)$ more general operator

$$Lf(x,\mu) = -i\mu\frac{\partial f}{\partial x} + \int_{-1}^{1} k(x,\mu,\mu')f(x,\mu')d\mu', (x,\mu)\in D, \qquad (40)$$

where

$$k(x,\mu,\mu') = \sum_{j=1}^{n} a_j(x)b_{1,j}(\mu)b_j(\mu'). \qquad (41)$$

We keep the same conditions (as for the operator (1)) on the coefficients, namely, for some constants $M>0, \varepsilon>0$, we suppose that $\left|a_j(x)\right| \le M\exp\left(-\varepsilon\left|x\right|\right), x\in R, j=1,...,n$ and the functions $b_{1,j}(\mu), b_j(\mu)$, $\mu\in[-1,1]$ admit holomorphic prolongation in the circle $\left|z\right| < 1+\varepsilon$.

We want to show that the operator L may be including in previous scheme. But we consider the spectrum of L only. We choose the factorization $a_j(x) = \overline{a_{1,j}(x)}a_{2,j}(x)$, $\left|a_{1,j}(x)\right| = \left|a_{2,j}(x)\right|$, which after Fourier transformation of the equality (40) gives Friedrichs' model

$$T = S + \sum_{j=1}^{n} A_j^* B_j \qquad (42)$$

in the same space H. Integral operators $A_j, B_j : H \to L^2(R)$ are

$$\begin{cases} A_j\varphi(x) = \dfrac{1}{\sqrt{2\pi}} a_{1,j}(x)\displaystyle\int_R\int_{-1}^{1}\overline{b_{1,j}(\mu)}\varphi(s,\mu)e^{ix\frac{s}{\mu}}\frac{1}{|\mu|}d\mu\, ds, x\in R, \\ B_j\varphi(x) = \dfrac{1}{\sqrt{2\pi}} a_{2,j}(x)\displaystyle\int_R\int_{-1}^{1} b_j(\mu)\varphi(s,\mu)e^{ix\frac{s}{\mu}}\frac{1}{|\mu|}d\mu\, ds. \end{cases}$$

$$K_{jm}(\zeta) = \delta_{jm} + B_j S_\zeta A_m^*, \operatorname{Im}\zeta \ne 0 \qquad (43)$$

and (compare with (15)) $l_{jm}(\tau,\zeta) = \int_{-1}^{1}\frac{b_j(\mu)b_{1,m}(\mu)}{\tau\mu-\zeta}d\mu.$ Obviously, theorem 2.3 holds for the operator (43) too. One can easily repeat the proof of Theorem 3.2 to obtain in the neighbourhood of $\zeta=0$ the decomposition

$$K_{jm}(\zeta) - \delta_{jm} = \frac{1}{2\pi} b_j(0)b_{1,m}(0)\gamma(\zeta)\left(\bullet, a_{1,m}\right)_{L^2(R)} a_{2,j} + Q_{jm}(\zeta), \qquad (44)$$

where the operator function $Q(\zeta)$ is holomorphic in some circle $\left|\zeta\right| < \delta$ and $\gamma(\zeta) = -\pi i\cdot\operatorname{signIm}\zeta\cdot\ln\zeta$. The function $\ln\zeta$ is continuous in the domain $\zeta\notin[0,\infty)$ and $\ln(-1) = \pi i$.

Let us rewrite the operator (42) under the form $T = S + A^{*}B$. We introduce direct sum $G = G_1 \oplus ... \oplus G_n$, where $G_j = L^2(R)$. Then we interpret the factors in (42) as operators $A_j, B_j : H \to G_j$ and introduce the following operators $A, B : H \to G$:

$$A\varphi = \begin{pmatrix} A_1\varphi \\ . \\ . \\ . \\ A_n\varphi \end{pmatrix}, \; B\varphi = \begin{pmatrix} B_1\varphi \\ . \\ . \\ . \\ B_n\varphi \end{pmatrix} \in G. \tag{45}$$

If $c \in G$ then

$$(A\varphi, c)_G = \sum_m (A_m\varphi, c_m)_{G_m} = \left(\varphi, \sum_m A_m^{*}, c_m \right)_H .$$

So,

$$A^{*}c = \sum_m A_m^{*} c_m. \tag{46}$$

Therefore $A^{*}B\varphi = \sum_j A_j^{*} B_j$ and $T = S + A^{*}B$ (see (42)). We have (see (45)—(46))

$$(BS_\zeta A^{*}c)_j = B_j S_\zeta A^{*}c = \sum_m B_j S_\zeta A_m^{*} c_m$$

i.e. we have matrix form $K(\zeta) = 1 + BS_\zeta A^{*} = (K_{jm}(\zeta))_{j,m=1}^{n}$ (see (43)).

Denote

$$\alpha(x) = \begin{pmatrix} \overline{b_{1,1}(0)} a_{1,1}(x) \\ . \\ . \\ . \\ \overline{b_{1,n}(0)} a_{1,n}(x) \end{pmatrix}, \; \beta(x) = \begin{pmatrix} b_1(0) a_{2,1}(x) \\ . \\ . \\ . \\ b_n(0) a_{2,n}(x) \end{pmatrix} \in G.$$

Then (see (44))

$$(K(\zeta) - 1)c = \frac{1}{2\pi} \gamma(\zeta)(c, \alpha)_G \beta + Q(\zeta)c,$$

where $(Q(\zeta)c)_j = \sum_m Q_{jm}(\zeta)\, c_m$. According to [5]

$$\left\| Q_{jm}(\zeta) \right\| \le N(\delta) \left\| b_j \right\|_{C^1} \left\| b_{1,m} \right\|_{C^1} \left\| a_{1,m} \right\|_{\delta}^{\frac{1}{2}} \left\| a_{1,j} \right\|_{\delta}^{\frac{1}{2}}, \tag{47}$$

where $N(\delta)$ is some given function on δ and $\|a\|_\delta^2 \equiv \int_R |a(x)|^2 e^{2\delta|x|}dx.$

Using the notation (see (41))

$$M(a,b)^2 \equiv \sum_j \|b_j\|_{C^1}^2 \|a_{2,j}\|_\delta = \sum_j \|b_j\|_{C^1}^2 \left(\int_R |a_j(x)| e^{2\delta|x|} dx \right)^{\frac{1}{2}}$$

$$M(a,b)^2 \equiv \sum_m \|b_{1,m}\|_{C^1}^2 \|a_{1,m}\| = \sum_m \|b_{1,m}\|_{C^1}^2 \left(\int_R |a_m(x)| e^{2\delta|x|} \right)^{\frac{1}{2}}$$

and the estimates (see(47)), we obtain $\|Q(\zeta)\| \le N(\delta)M(a,b)M_1(a,b).$ We also need the value (recall that $a_{2,j}(x)\overline{a_{1,j}(x)} = a_j(x)$)

$$(\beta,\alpha)_G = \sum_j b_j(0)b_{1,j}(0)\int_R a_j(x)dx.$$

By analogy to the work [5], we obtain the following Theorem.

5.1. Theorem. Under one of the following conditions

1) $(\beta,\alpha)_G = 0,\ N(\delta)M(a,b)M_1(a,b) < 1$

or

2) $(\beta,\alpha)_G \ne 0,\ N(\delta)\|\alpha\|_G \|\beta\|_G M(a,b)M_1(a,b) < |(\beta,\alpha)_G|$

the operator L (see(40)) has not point spectrum in the circle $|\zeta| < \delta$.

Note, that exponential decreasing of the functions $a_j(x), |x| \to \infty$ permit (again by analogy to work [5]) to prove the analogy of Theorem 2.3. Taking into account Theorem 5.1, we obtain that point spectrum of the operator L is finite.

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Автори розглядають у просторі $L^2(D), D = R \times [-1,1]$ транспортний оператор

$$Lf = -i\mu \frac{\partial f}{\partial x} + a(x)b_1(\mu)\int_{-1}^{1} b(\mu')f(x,\mu')d\mu'.$$

Щоб отримати вигляд розвязку рівняння $\dot{u} = iLu, u\big|_{t=0} = u(0)$ автори вводять деякий інтеграл (у вигляді відомого виразу резольвенти через півгрупу) і доводять прямо, що цей інтеграл є відповідною півгрупою. Щоб спростити обчислення, автори зводять оператор L до деякої моделі Фрідріхса, використовуючи перетворення Фур'є.

Ключові слова: *спектр ,транспортний оператор, модель Фрідріхса, півгрупа.*

Отримано 16.10.2010

УДК 519.21

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СУЩЕСТВОВАНИЕ ВТОРОГО МОМЕНТА РЕШЕНИЯ ЛИНЕЙНОГО СТОХАСТИЧЕСКОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ С МАРКОВСКИМИ ВОЗМУЩЕНИЯМИ И ЕГО ПОВЕДЕНИЕ НА БЕСКОНЕЧНОСТИ

Для стохастической задачи Коши линейного уравнения в частных производных с непрерывным марковским процессом доказано существование решения в среднем квадратическом, получены достаточные условия асимптотической устойчивости в среднем квадратическом решения этой задачи.

Ключевые слова: *задача Коши, стохастические дифференциальное уравнение, уравнение в частных производных, асимптотическая устойчивость, устойчивость в среднем квадратическом, марковский процесс.*

Введение. Доказательству существования и асимптотического поведения решений детерминированных уравнений в частных производных посвящено достаточное число монографий и статей, которые можно найти в монографиях [13], [15], [22].

Когда было введено понятие стохастического дифференциала и интеграла, как функции верхнего предела, замены переменных Ито для стохастического дифференциала, введения понятия стохастиче-