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I. M. Konet, Doctor of Phys. and Mathem. Sciences, Professor,
T. M. Pylypiuk, teacher

Kamianets-Podilsky National Ivan Ohienko University,
Kamianets-Podilsky

THE INTEGRAL REPRESENTATION OF THE SOLUTION OF MIXED PARABOLIC PROBLEM IN PIECE-HOMOGENEOUS POLAR AXIS WITH SOFT LIMITS

By means of method of hybrid integral transform of Legendre-Bessel-Legendre type with the spectral parameter the integral representation of exact analytical solution of mixed problem for the system of equations of parabolic type in the piece-homogeneous polar axis $r \ge R_0 > 0$ with soft limits is obtained. Modeling of evolutionary process is made by the method of hybrid differential Legendre-Bessel-Legendre operator.

Keywords: *parabolic equation, initial and boundary conditions, conjugate conditions, the differential operator, hybrid integral transform, Green functions, the influence functions.*

Introduction. The theory of boundary value problems for partial differential equations is an important section of the modern theory of differential equations which is intensively developing in the present time. The topicality of the problem is determined by significance of its results for development of many mathematical issues as well as by numerous applications of its achievements in mathematic modeling of different processes and phenomenon of physics, chemistry, mechanics, biology, medicine, economics, engineering.

It is well known that the complexity of the investigated boundary-value problems significantly depends on the coefficients of equations (different types of degeneracy and features) and the geometry of domain (smoothness of the boundary, the presence of corner points, etc.) in which the problem is considered. Presently the properties of solutions of boundary value problems for linear, quasi-linear, and certain classes of nonlinear equations in homogeneous domains which are stipulated by the above-mentioned properties of the coefficients of equations and geometry of domain are studied in detail, functional spaces of correctness of problems for some domains are constructed [1; 2].

However, many important applied problems of thermophysics, thermodynamics, theory of elasticity, theory of electrical circuits, theory of vibrations lead to boundary value problems for partial differential equations not only in homogeneous domains, if the coefficients of the equations are continuous, but also in inhomogeneous and piece-homogeneous domains if the coefficients of the equations are piece-continuous or in particular piece-constant [3; 4].

Besides the method of separation of variables [5], which is one of the important and effective methods of learning the linear boundary value problems for partial differential equations there is the method of integral transforms, which makes it possible to obtain in analytical form solutions of some boundary-value problems via their integral images. It is also worth mentioning that for a rather wide class of problems (in piece-homogeneous domains) there proved to be effective the method of hybrid integral transforms which are generated by differential operators if on each of components of piece-homogeneous domain there are considered different differential operators or the differential operators of the same form, but with different sets of coefficients [6–9].

In theoretical investigations and applied problems the most frequently used differential operators of the second order, in particular Fourier differential operator $F = \frac{d^2}{dr^2}$, Euler differential operator

$$B_\alpha^* = r^2\frac{d^2}{dr^2} + (2\alpha+1)r\frac{d}{dr} + \alpha^2,$$

Bessel differential operator

$$B_{\upsilon,\alpha} = \frac{d^2}{dr^2} + \frac{2\alpha+1}{r}\frac{d}{dr} - \frac{\upsilon^2-\alpha^2}{r^2},$$

Legendre differential operator

$$\Lambda_{(\mu)} = \frac{d^2}{dr^2} + cthr\frac{d}{dr} + \frac{1}{4} + \frac{1}{2}\left(\frac{\mu_1^2}{1-chr} + \frac{\mu_2^2}{1+chr}\right)$$

and Kontorovich-Lebedev differential operator

$$B_\alpha = r^2\frac{d^2}{dr^2} + (2\alpha+1)r\frac{d}{dr} + \alpha^2 - \lambda^2 r^2.$$

If $\theta(x)$ is the Heaviside step function and L_j is one of listed differential operators, then we can always create hybrid differential operator that corresponds to the geometric structure of piece-homogeneous domain.

For example, for the piece-homogeneous interval $(R_0, R_1)\cup\cup(R_1, R_2)\cup(R_2, R_3)$ it is possible to create hybrid differential operator

$$M = \theta(r-R_0)\theta(R_1-r)a_1^2L_1 + \theta(r-R_1)\theta(R_2-r)a_2^2L_2 + \\ +\theta(r-R_2)\theta(R_3-r)a_3^2L_3;\ a_j^2 = const.$$

It is obvious that operator L_1 is defined in the interval (R_0, R_1), operator L_2 is defined in the interval (R_1, R_2) and operator L_3 is defined in the interval (R_2, R_3). It is clear, that if we change the order of operators L_1, L_2, L_3 we get other hybrid differential operator.

In this article we propose the integral representation of exact analytical solution of mixed problem for the system of evolution equations of parabolic type modeling by the method of hybrid differential Legendre-Bessel-Legendre operator in the piece-homogeneous polar axis $(R_0, R_1) \cup \cup (R_1, R_2) \cup (R_2, +\infty)$ with soft limits.

Formulation of the problem. There is considered the problem of structure of solution which is bounded in the set

$$D_2^+ = \left\{(t,r) : t \in (0,+\infty); r \in I_2^+ = (R_0, R_1) \cup (R_1, R_2) \cup (R_2, +\infty), R_0 > 0\right\}$$

for the system of partial differential equations of parabolic type [5].

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + \gamma_1^2 u_1 - a_1^2 \Lambda_{(\mu)_1}[u_1] = f_1(t,r),\ r \in (R_0, R_1), \\ &\frac{\partial u_2}{\partial t} + \gamma_2^2 u_2 - a_2^2 B_{\upsilon,\alpha}[u_2] = f_2(t,r),\ r \in (R_1, R_2), \\ &\frac{\partial u_3}{\partial t} + \gamma_3^2 u_3 - a_3^2 \Lambda_{(\mu)_2}[u_3] = f_3(t,r),\ r \in (R_2, +\infty) \end{aligned} \tag{1}$$

with initial conditions

$$\begin{aligned} &u_1(t,r)\big|_{t=0} = g_1(r), r \in (R_0, R_1); \\ &u_2(t,r)\big|_{t=0} = g_2(r), r \in (R_1, R_2); \\ &u_3(t,r)\big|_{t=0} = g_3(r), r \in (R_2, +\infty); \end{aligned} \tag{2}$$

boundary conditions

$$L_{11}^0\left[u_1(t,r)\right]\big|_{r=R_0} = g_0(t), \ \lim_{r\to+\infty}\left[shr\frac{\partial u_3}{\partial r}\right] = 0 \tag{3}$$

and conjugate conditions

$$\left(L_{j1}^k\left[u_k(t,r)\right] - L_{j2}^k\left[u_{k+1}(t,r)\right]\right)\Big|_{r=R_k} = \omega_{jk}(t); \quad j,k = 1,2. \tag{4}$$

Bessel differential operator $B_{\upsilon,\alpha} = \frac{\partial^2}{\partial r^2} + (2\alpha+1)r^{-1}\frac{\partial}{\partial r} - (\upsilon^2 - \alpha^2)r^{-2}$ [10] and Legendre differential operators $\Lambda_{(\mu)_j} = \frac{\partial^2}{\partial r^2} + cthr\frac{\partial}{\partial r} + \frac{1}{4} + \frac{1}{2}\left[\mu_{1j}^2(1-chr)^{-1} + \mu_{2j}^2(1+chr)^{-1}\right]$ [11] take part in system (1); $\upsilon \ge \alpha$, $2\alpha + 1 > 0$, $\mu_{1j} \ge \mu_{2j} \ge 0$, $(\mu) = ((\mu)_1; (\mu)_2)$.

In the boundary condition and conjugate conditions differential operators $L_{jm}^{i} = \left(\alpha_{jm}^{i} + \delta_{jm}^{i} \frac{\partial}{\partial t} \right) \frac{\partial}{\partial r} + \beta_{jm}^{i} + \gamma_{jm}^{i} \frac{\partial}{\partial t}; \quad j, m = 1, 2; \quad i = 0, 1, 2$ take part.

We consider, that conditions on the coefficients are true: $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $\gamma_1^2 \geq 0$, $\gamma_2^2 \geq 0$, $\gamma_3^2 \geq 0$; $|\alpha_{11}^0| + \beta_{11}^0 \neq 0$, $\alpha_{11}^0 \leq 0$, $\beta_{11}^0 \geq 0$; $\alpha_{jm}^k \geq 0$, $\beta_{jm}^k \geq 0$, $\delta_{jm}^k \geq 0$, $\gamma_{jm}^k \geq 0$; $c_{j1,k} = \alpha_{2j}^k \beta_{1j}^k - \alpha_{1j}^k \beta_{2j}^k$; $c_{j2,k} \equiv \delta_{2j}^k \gamma_{1j}^k - \delta_{1j}^k \gamma_{2j}^k = 0$; $c_{11,k} \cdot c_{21,k} > 0$; $\alpha_{11}^k \gamma_{21}^k - \alpha_{21}^k \gamma_{11}^k = \beta_{11}^k \delta_{21}^k - \beta_{21}^k \delta_{11}^k$, $\alpha_{12}^k \gamma_{22}^k - \alpha_{22}^k \gamma_{12}^k = \beta_{12}^k \delta_{22}^k - \beta_{22}^k \delta_{12}^k$.

Remark 1. The presence of the differentiation operator in time $\frac{\partial}{\partial t}$ in the boundary condition in the point $r = R_0$ and in the conjugate conditions we interpret on the base of physical reasons about heat waves as softness of boundary of domain on reflection of waves.

Remark 2. In the case of hard boundary of domain on reflection of waves ($\delta_{11}^0 = 0$, $\gamma_{11}^0 = 0$, $\delta_{jm}^k = 0$, $\gamma_{jm}^k = 0$), we have mixed problem with classic boundary condition and classic conjugate conditions, solution of which is obtained from the solution of the problem (1)–(4) as a particular case.

The main part. Let's construct the solution of parabolic problem of conjugation (1)–(4) by the method of hybrid integral transform with the spectral parameter generated in the set I_2^+ by hybrid differential operator (GDO)

$$M_{\upsilon,\alpha}^{(\mu)} = \theta(r - R_0)\theta(R_1 - r)a_1^2 \Lambda_{(\mu)_1} + \\ + \theta(r - R_1)\theta(R_2 - r)a_2^2 B_{\upsilon,\alpha} + \theta(r - R_2)\Lambda_{(\mu)_2}. \tag{5}$$

Definition. The domain of definition of the HDO $M_{\upsilon,\alpha}^{(\mu)}$ we call the set *G* of functions $g(r) = \{g_1(r); g_2(r); g_3(r)\}$ with such properties:

1) the function $f(r) = \left\{ \Lambda_{(\mu)_1}[g_1(r)]; B_{\upsilon,\alpha}[g_2(r)]; \Lambda_{(\mu)_2}[g_3(r)] \right\}$ is continuous in the set I_2^+;
2) functions $g_j(r)$ satisfy the boundary conditions

$$\left(\tilde{\alpha}_{11}^0 \frac{d}{dr} + \tilde{\beta}_{11}^0 \right) g_1(r) \Big|_{r=R_0} = 0, \quad \lim_{r \to +\infty} \left[\sqrt{shr}\, g_3'(r) \right] = 0 \tag{6}$$

and conjugate conditions

$$\left[\left(\tilde{\alpha}_{j1}^k \frac{d}{dr} + \tilde{\beta}_{j1}^k \right) g_k(r) - \left(\tilde{\alpha}_{j2}^k \frac{d}{dr} + \tilde{\beta}_{j2}^k \right) g_{k+1}(r) \right] \Big|_{r=R_k} = 0, \; j, k = 1, 2. \tag{7}$$

Functions of the spectral parameter β: $\tilde{\alpha}_{jm}^{i} = \alpha_{jm}^{i} - (\beta^2 + \gamma^2)\delta_{jm}^{i}$, $\tilde{\beta}_{jm}^{i} = \beta_{jm}^{i} - (\beta^2 + \gamma^2)\gamma_{jm}^{i}$, $j, m = 1,2$, $i = 0,1,2$ take part in the equations (6), (7).

Let's consider the functions $u(r) = \{u_1(r); u_2(r); u_3(r)\} \in G$ and $v(r) = \{v_1(r); v_2(r); v_3(r)\} \in G$.

From the algebraic system

$$\left(\tilde{\alpha}_{j1}^{k}\frac{d}{dr} + \tilde{\beta}_{j1}^{k}\right)u_k(r)\Big|_{r=R_k} = \left(\tilde{\alpha}_{j2}^{k}\frac{d}{dr} + \tilde{\beta}_{j2}^{k}\right)u_{k+1}(r)\Big|_{r=R_k}$$

we find the functional dependence:

$$u'_k(R_k) = \frac{1}{c_{11,k}}\left[\tilde{a}_{21}^{k}u'_{k+1}(R_k) + \tilde{a}_{22}^{k}u_{k+1}(R_k)\right],$$

$$u_k(R_k) = -\frac{1}{c_{11,k}}\left[\tilde{a}_{11}^{k}u'_{k+1}(R_k) + \tilde{a}_{12}^{k}u_{k+1}(R_k)\right].$$

The same equalities connect the components of function $v(r)$:

$$v'_k(R_k) = c_{11,k}^{-1}[\tilde{a}_{21}^{k}v'_{k+1}(R_k) + \tilde{a}_{22}^{k}v_{k+1}(R_k)],$$

$$v_k(R_k) = -c_{11,k}^{-1}[\tilde{a}_{11}^{k}v'_{k+1}(R_k) + \tilde{a}_{12}^{k}v_{k+1}(R_k)].$$

We accept the denotation:

$$\tilde{a}_{11}^{k} = \tilde{\alpha}_{11}^{k}\tilde{\alpha}_{22}^{k} - \tilde{\alpha}_{21}^{k}\tilde{\alpha}_{12}^{k},\ \tilde{a}_{12}^{k} = \tilde{\alpha}_{11}^{k}\tilde{\beta}_{22}^{k} - \tilde{\alpha}_{21}^{k}\tilde{\beta}_{12}^{k},$$

$$\tilde{a}_{21}^{k} = \tilde{\beta}_{11}^{k}\tilde{\alpha}_{22}^{k} - \tilde{\beta}_{21}^{k}\tilde{\alpha}_{12}^{k},\ \tilde{a}_{22}^{k} = \tilde{\beta}_{11}^{k}\tilde{\beta}_{22}^{k} - \tilde{\beta}_{21}^{k}\tilde{\beta}_{12}^{k}.$$

Immediately establish that

$$\begin{aligned}&\left[u'_k(r)v_k(r) - u_k(r)v'_k(r)\right]\Big|_{r=R_k} = \\ &= c_{11,k}^{-2}\left(\tilde{a}_{11}^{k}\tilde{a}_{22}^{k} - \tilde{a}_{12}^{k}\tilde{a}_{21}^{k}\right)\left(u'_{k+1}(r)v_{k+1}(r) - \right. \\ &\left. -u_{k+1}(r)v'_{k+1}(r)\right)\Big|_{r=R_k} = c_{11,k}^{-1}c_{21,k}\left(u'_{k+1}(R_k)v_{k+1}(R_k) - \right. \\ &\left. -u_{k+1}(R_k)v'_{k+1}(R_k)\right).\end{aligned} \tag{8}$$

Let's define the values

$$a_1^2\sigma_1 = \frac{c_{11,1}c_{11,2}}{c_{21,1}c_{21,2}}\frac{shR_2}{shR_1}\frac{R_1^{2\alpha+1}}{R_2^{2\alpha+1}},\ a_2^2\sigma_2 = \frac{c_{11,2}}{c_{21,2}}\frac{shR_2}{R_2^{2\alpha+1}},\ a_3^2\sigma_3 = 1,$$

weight function

$$\begin{aligned}\sigma(r) &= \theta(r - R_0)\theta(R_1 - r)\sigma_1 shr + \\ &+\theta(r - R_1)\theta(R_2 - r)\sigma_2 r^{2\alpha+1} + \theta(r - R_2)\sigma_3 shr\end{aligned} \tag{9}$$

and scalar product

$$\big(u(r), v(r)\big) = \int_{R_0}^{+\infty} u(r)v(r)\sigma(r)dr \equiv \int_{R_0}^{R_1} u_1(r)v_1(r)\sigma_1 shrdr + \\ + \int_{R_1}^{R_2} u_2(r)v_2(r)\sigma_2 r^{2\alpha+1}dr + \int_{R_2}^{+\infty} u_3(r)v_3(r)\sigma_3 shrdr. \tag{10}$$

Lemma. HDO $M_{\upsilon,\alpha}^{(\mu)}$, defined in equality (5) is a self-conjugate operator.

Proof. Let's consider scalar product

$$\Big(M_{\upsilon,\alpha}^{(\mu)}[u], v(r)\Big) = \int_{R_0}^{+\infty} M_{\upsilon,\alpha}^{(\mu)}[u]v(r)\sigma(r)dr = \\ = \int_{R_0}^{R_1} a_1^2 \Lambda_{(\mu)_1}[u_1(r)]v_1(r)\sigma_1 shrdr + \\ + \int_{R_1}^{R_2} a_2^2 B_{\upsilon,\alpha}[u_2(r)]v_2(r)\sigma_2 r^{2\alpha+1}dr + \int_{R_2}^{+\infty} a_3^2 \Lambda_{(\mu)_2}[u_3(r)]v_3(r)\sigma_3 shrdr. \tag{11}$$

Let's integrate in (11) by parts twice. We get:

$$\Big(M_{\upsilon,\alpha}^{(\mu)}[u], v\Big) = \left[a_1^2\sigma_1 shr\left(\frac{du_1}{dr}v_1(r) - u_1(r)\frac{dv_1}{dr}\right)\right]\Bigg|_{R_0}^{R_1} + \\ + \int_{R_0}^{R_1} u_1(r)\Big(a_1^2\Lambda_{(\mu)_1}[v_1]\Big)\times\sigma_1 shrdr + \\ + \left[a_2^2\sigma_2 r^{2\alpha+1}\left(\frac{du_2}{dr}v_2 - u_2\frac{dv_2}{dr}\right)\right]\Bigg|_{R_1}^{R_2} + \int_{R_1}^{R_2} u_2(r)\Big(a_2^2 B_{\upsilon,\alpha}[v_2]\Big)\times \\ \times\sigma_2 r^{2\alpha+1}dr + \left[a_3^2\sigma_3 shr\left(\frac{du_3}{dr}v_3 - u_3\frac{dv_3}{dr}\right)\right]\Bigg|_{R_2}^{+\infty} + \\ + \int_{R_2}^{+\infty} u_3(r)\Big(a_3^2\Lambda_{(\mu)_2}[v_3]\Big)\sigma_3 shrdr. \tag{12}$$

If $\tilde{\alpha}_{11}^0 \neq 0$, then because of the boundary condition in the point $r = R_0$

$$\left(\frac{du_1}{dr}v_1 - u_1\frac{dv_1}{dr}\right)\Bigg|_{r=R_0} = \frac{1}{\tilde{\alpha}_{11}^0}\left[\left(\tilde{\alpha}_{11}^0\frac{du_1}{dr} + \tilde{\beta}_{11}^0 u_1\right)v_1\right]\Bigg|_{r=R_0} - \\ - \frac{\tilde{\beta}_{11}^0}{\tilde{\alpha}_{11}^0}u_1(R_0)v_1(R_0) - u_1(R_0)\frac{dv_1}{dr}\Bigg|_{r=R_0} = \\ = (\tilde{\alpha}_{11}^0)^{-1}0\cdot v_1(R_0) - (\tilde{\alpha}_{11}^0)^{-1}v_1(R_0)\cdot 0 = 0. \tag{13}$$

Because of the limited conditions (6)

$$\lim_{r\to+\infty} shr\left(\frac{du_3}{dr}v_3 - u_3\frac{dv_3}{dr}\right) = \\ = \lim_{r\to+\infty}\left[\sqrt{shr}\frac{du_3}{dr}(\sqrt{shr}v_3) - (\sqrt{shr}u_3)\sqrt{shr}\frac{dv_3}{dr}\right] = 0. \tag{14}$$

In the point $r = R_1$ because of the basic identity (8) at $k = 1$ we have, that

$$a_1^2\sigma_1 shR_1\left(u_1'v_1 - u_1v_1'\right)\Big|_{r=R_1} - a_2^2\sigma_2 R_1^{2\alpha+1}\left(u_2'v_2 - u_2v_2'\right)\Big|_{r=R_1} = \\ = \left(a_1^2\sigma_1 shR_1\frac{c_{21,1}}{c_{11,1}} - a_2^2\sigma_2 R_1^{2\alpha+1}\right)\left(u_2'v_2 - u_2v_2'\right)\Big|_{r=R_1} = 0, \tag{15}$$

because of choosing numbers σ_1 and σ_2

$$a_1^2\sigma_1 shR_1\frac{c_{21,1}}{c_{11,1}} - a_2^2\sigma_2 R_1^{2\alpha+1} = \frac{c_{11,1}c_{11,2}}{c_{21,1}c_{21,2}}\frac{shR_2}{R_2^{2\alpha+1}}R_1^{2\alpha+1}\frac{c_{21,1}}{c_{11,1}} - \\ - \frac{c_{11,2}}{c_{21,2}}\frac{shR_2}{R_2^{2\alpha+1}}R_1^{2\alpha+1} = \frac{c_{11,2}}{c_{21,2}}\frac{shR_2}{R_2^{2\alpha+1}}R_1^{2\alpha+1}(1-1) = 0\ .$$

In the point $r = R_2$ because of the basic identity (8) at $k = 2$ we have, that

$$a_2^2\sigma_2 R_2^{2\alpha+1}\left(u_2'v_2 - u_2v_2'\right)\Big|_{r=R_2} - a_3^2\sigma_3 shR_2\left(u_3'v_3 - u_3v_3'\right)\Big|_{r=R_2} = \\ = \left(a_2^2\sigma_2 R_2^{2\alpha+1}\frac{c_{21,2}}{c_{11,2}} - a_3^2\sigma_3 shR_2\right)\left(u_3'v_3 - u_3v_3'\right)\Big|_{r=R_2} = 0, \tag{16}$$

because of choosing numbers σ_2 and σ_3

$$a_2^2\sigma_2 R_2^{2\alpha+1}\frac{c_{21,2}}{c_{11,2}} - a_3^2\sigma_3 shR_2 = \frac{c_{11,2}}{c_{21,2}}shR_2\frac{c_{21,2}}{c_{11,2}} - shR_2 = shR_2(1-1) = 0.$$

In the equation (12) outside the integral terms are equal to zero because of equalities (13)–(16). The equality (12) takes the form:

$$\left(M_{\upsilon,\alpha}^{(\mu)}[u(r)], v(r)\right) = \left(u(r), M_{\upsilon,\alpha}^{(\mu)}[v(r)]\right).$$

So HDO $M_{\upsilon,\alpha}^{(\mu)}$ is self-conjugate. The lemma is proved.

Conclusion: eigenvalues of HDO $M_{\upsilon,\alpha}^{(\mu)}$ are real. Since the HDO $M_{\upsilon,\alpha}^{(\mu)}$ has one singular point $r = +\infty$, then it's spectrum is continuous [10]. We can assume, that spectral parameter $\beta \in (0, +\infty)$.

Let's assume that spectral function

$$V_{\upsilon,\alpha}^{(\mu)}(r,\beta)=\sum_{k=1}^{2}\theta(r-R_{k-1})\theta(R_k-r)V_{\upsilon,\alpha;k}^{(\mu)}(r,\beta)+\theta(r-R_2)V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta) \quad (17)$$

corresponds to spectral parameter β.

Then functions $V_{\upsilon,\alpha;k}^{(\mu)}(r,\beta)$ must satisfy respectively differential equations

$$\left(a_1^2\Lambda_{(\mu)_1}+b_1^2\right)V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta)=0,\ r\in(R_0,R_1),$$
$$\left(a_2^2B_{\upsilon,\alpha}+b_2^2\right)V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta)=0,\ r\in(R_1,R_2), \quad (18)$$
$$\left(a_3^2\Lambda_{(\mu)_2}+b_3^2\right)V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)=0,\ r\in(R_2,+\infty)$$

boundary conditions (6) and conjugate conditions (7); $b_j^2=\beta^2+k_j^2$, $k_j^2\ge 0$.

The fundamental system of solutions for Legendre differential equation $\left(\Lambda_{(\mu)_1}+\overline{b}_1^2\right)v=0$ is formed by generalized attached Legendre first order functions $A_{\upsilon_1^*}^{(\mu)_1}(chr)$ and $B_{\upsilon_1^*}^{(\mu)_1}(chr)$ [11], $\upsilon_1^*=-1/2+i\overline{b}_1$; the fundamental system of solutions for Bessel differential equation $\left(B_{\upsilon,\alpha}+\overline{b}_2^2\right)v=0$ is formed by Bessel functions $J_{\upsilon,\alpha}(\overline{b}_2r)$ та $N_{\upsilon,\alpha}(\overline{b}_2r)$ [10]; the fundamental system of solutions for Legendre differential equation $\left(\Lambda_{(\mu)_2}+\overline{b}_3^2\right)v=0$ is formed by functions $A_{\upsilon_3^*}^{(\mu)_2}(chr)$ and $B_{\upsilon_3^*}^{(\mu)_2}(chr)$ [11], $\upsilon_3^*=-1/2+i\overline{b}_3$; $\overline{b}_j=a_j^{-1}b_j\equiv a_j^{-1}(\beta^2+k_j^2)$, $k_j^2\ge 0$, $j=\overline{1,3}$.

If by virtue of linearity of spectral problem put

$$V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta)=A_1A_{\upsilon_1^*}^{(\mu)_1}(chr)+B_1B_{\upsilon_1^*}^{(\mu)_1}(chr),\quad r\in(R_0,R_1),$$
$$V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta)=A_2J_{\upsilon,\alpha}(\overline{b}_2r)+B_2N_{\upsilon,\alpha}(\overline{b}_2r),\quad r\in(R_1,R_2), \quad (19)$$
$$V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)=A_3A_{\upsilon_3^*}^{(\mu)_2}(chr)+B_3B_{\upsilon_3^*}^{(\mu)_2}(chr),\quad r\in(R_2,+\infty),$$

then we have homogeneous algebraic system of five equations for the determination of the six variables:

$$Y_{\upsilon_1^*;11}^{(\mu)_1;01}(chR_0)A_1+Y_{\upsilon_1^*;11}^{(\mu)_1;02}(chR_0)B_1=0;$$
$$Y_{\upsilon_1^*;j1}^{(\mu)_1;11}(chR_1)A_1+Y_{\upsilon_1^*;j1}^{(\mu)_1;12}(chR_1)B_1-u_{\upsilon,\alpha;j2}^{11}(\overline{b}_2R_1)A_2-$$
$$-u_{\upsilon,\alpha;j2}^{12}(\overline{b}_2R_1)B_2=0,\ \ u_{\upsilon,\alpha;j1}^{21}(\overline{b}_2R_2)A_2+u_{\upsilon,\alpha;j1}^{22}(\overline{b}_2R_2)B_2- \quad (20)$$
$$-Y_{\upsilon_3^*;j2}^{(\mu)_2;21}(chR_2)A_3-Y_{\upsilon_3^*;j2}^{(\mu)_2;22}(chR_2)B_3=0.$$

Let's introduce the functions into consideration:

$$q_\alpha(\beta)=\frac{2}{\pi}\frac{c_{21,1}}{\bar b_2^{2\alpha}R_1^{2\alpha+1}},\quad q_{(\mu)_2}(\beta)=\frac{c_{21,2}}{S_{(\mu)_2}(\bar b_3)shR_2},\quad \delta_{(\mu)_1;j1}(chR_0,chR_1)=$$

$$=Y^{(\mu)_1;01}_{\upsilon_1^*;11}(chR_0)Y^{(\mu)_1;12}_{\upsilon_1^*;j1}(chR_1)-Y^{(\mu)_1;02}_{\upsilon_1^*;11}(chR_0)Y^{(\mu)_1;11}_{\upsilon_1^*;j1}(chR_1);$$

$$\delta_{\upsilon,\alpha;jk}(\bar b_2R_1,\bar b_2R_2)=u^{11}_{\upsilon,\alpha;j2}(\bar b_2R_1)u^{22}_{\nu,\alpha;k1}(\bar b_2R_2)-$$

$$-u^{12}_{\upsilon,\alpha;j2}(\bar b_2R_1)u^{21}_{\nu,\alpha;k1}(\bar b_2R_2);\quad a^{(\mu)_1}_{\upsilon,\alpha;j}(\beta)=\delta_{\nu,\alpha;1j}(\bar b_2R_1,\bar b_2R_2)\times$$

$$\times\delta_{(\mu)_1;21}(chR_0,chR_1)-\delta_{\upsilon,\alpha;2j}(\bar b_2R_1,\bar b_2R_2)\times$$

$$\times\delta_{(\mu)_1;11}(chR_0,chR_1);\quad j=1,2;\quad \psi^1_{\upsilon,\alpha;j2}(\bar b_2R_1,\bar b_2r)=u^{11}_{\upsilon,\alpha;j2}(\bar b_2R_1)\times$$

$$\times N_{\upsilon,\alpha}(\bar b_2r)-u^{12}_{\upsilon,\alpha;j2}(\bar b_2R_1)J_{\upsilon,\alpha}(\bar b_2r),\quad j=1,2\,.$$

As a result of the standard solution [12] of algebraic system (20) and substituting the obtained values of variables A_j, B_j in the equality (19) we obtain the components $V^{(\mu)}_{\upsilon,\alpha;k}(r,\beta)$ of spectral function $V^{(\mu)}_{\upsilon,\alpha}(r,\beta)$ of HDO $M^{(\mu)}_{\upsilon,\alpha}$:

$$V^{(\mu)}_{\upsilon,\alpha;1}(r,\beta)=q_\alpha(\beta)q_{(\mu)_2}(\beta)[Y^{(\mu)_1;01}_{\upsilon_1^*;11}(chR_0)B^{(\mu)_1}_{\upsilon_1^*}(chr)-Y^{(\mu)_1;02}_{\upsilon_1^*;11}(chR_0)A^{(\mu)_1}_{\upsilon_1^*}(chr)]$$

$$\begin{aligned}V^{(\mu)}_{\upsilon,\alpha;2}(r,\beta)&=q_{(\mu)_2}(\beta)[\delta_{(\mu)_1;21}(chR_0,chR_1)\psi^1_{\upsilon,\alpha;12}(\bar b_2R_1,\bar b_2r)-\\&-\delta_{(\mu)_1;11}(chR_0,chR_1)\psi^1_{\upsilon,\alpha;22}(\bar b_2R_1,\bar b_2r)],\end{aligned}\tag{21}$$

$$V^{(\mu)}_{\upsilon,\alpha;3}(r,\beta)=\omega^{(\mu)}_{\upsilon,\alpha;1}(\beta)B^{(\mu)_2}_{\upsilon_3^*}(chr)-\omega^{(\mu)}_{\upsilon,\alpha;2}(\beta)A^{(\mu)_2}_{\upsilon_3^*}(chr)\,.$$

According to the equality (17) spectral function $V^{(\mu)}_{\upsilon,\alpha}(r,\beta)$ becomes defined.

Let's define the spectral density by the formula

$$\Omega^{(\mu)}_{\upsilon,\alpha}(\beta)=\frac{\beta\gamma_{(\mu)_2}(\bar b_3)S_{(\mu)_2}(\bar b_3)}{\left[\omega^{(\mu)}_{\upsilon,\alpha;1}(\beta)\right]^2+\left[\gamma_{(\mu)_2}(\bar b_3)\omega^{(\mu)}_{\upsilon,\alpha;2}(\beta)\right]^2}\,.\tag{22}$$

The presence of weight function $\sigma(r)$, spectral function $V^{(\mu)}_{\upsilon,\alpha}(r,\beta)$ and spectral density $\Omega^{(\mu)}_{\upsilon,\alpha}(\beta)$ makes it possible to determine the direct $H^{(\mu)}_{\upsilon,\alpha}$ and inverse $H^{-(\mu)}_{\upsilon,\alpha}$ hybrid integral transform (HIT) with the spectral parameter, generated in the set I_2^+ of HDO $M^{(\mu)}_{\upsilon,\alpha}$:

$$H^{(\mu)}_{\upsilon,\alpha}[g(r)]=\int\limits_{R_0}^{+\infty}g(r)V^{(\mu)}_{\upsilon,\alpha}(r,\beta)\sigma(r)dr\equiv\tilde g(\beta)\,,\tag{23}$$

$$H_{\upsilon,\alpha}^{-(\mu)}[\tilde{g}(\beta)] = \frac{2}{\pi}\int\limits_0^{+\infty} \tilde{g}(\beta)V_{\upsilon,\alpha}^{(\mu)}(r,\beta)\Omega_{\upsilon,\alpha}^{(\mu)}(\beta)d\beta \equiv g(r). \qquad (24)$$

Next statement is a mathematical justification of formulas (23), (24).

Theorem 1. If the function $f(r) = [\theta(r-R_0)\theta(R_1-r)\sqrt{shr} + \theta(r-R_1)\theta(R_2-r)r^{\alpha+1/2} + \theta(r-R_2)\sqrt{shr}]g(r)$ is continuous, absolutely summable and has bounded variation in the set $(R_0,+\infty)$, then for any $r \in I_2^+$ integral representation is true:

$$g(r) = \frac{2}{\pi}\int\limits_0^{\infty} V_{\upsilon,\alpha}^{(\mu)}(r,\beta)\int\limits_{R_0}^{\infty} g(\rho)V_{\upsilon,\alpha}^{(\mu)}(\rho,\beta)\sigma(\rho)d\rho\Omega_{\upsilon,\alpha}^{(\mu)}(\beta)d\beta. \qquad (25)$$

The proof of equality (25) is performed by the method of delta-shaped sequence (Cauchy kernel or Dirihle kernel) [10].

In basis of application HIT (23), (24) to the solution of appropriate boundary problems is the basic identity for the integral transform of the HDO $M_{\upsilon,\alpha}^{(\mu)}$.

Let's introduce values and functions into consideration:

$$d_1 = \frac{a_1^2\sigma_1 shR_1}{c_{11,1}}, \quad d_2 = \frac{a_2^2\sigma_2 R_2^{2\alpha+1}}{c_{11,2}}, \quad \tilde{g}_1(\beta) = \int\limits_{R_0}^{R_1} g_1(r)V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta)\sigma_1 shrdr,$$

$$\tilde{g}_2(\beta) = \int\limits_{R_1}^{R_2} g_2(r)V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta)\sigma_2 r^{2\alpha+1}dr,$$

$$\tilde{g}_3(\beta) = \int\limits_{R_2}^{+\infty} g_3(r)V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)\sigma_3 shrdr,$$

$$Z_{\upsilon,\alpha;i2}^{(\mu),k}(\beta) = \left(\tilde{\alpha}_{i2}^k\frac{d}{dr} + \tilde{\beta}_{i2}^k\right)V_{\upsilon,\alpha;k+1}^{(\mu)}(r,\beta)\Big|_{r=R_k}, \quad i,k = 1,2.$$

Theorem 2. If the function $f(r) = \{\Lambda_{(\mu)_1}[g_1(r)]; B_{\upsilon,\alpha}[g_2(r)]; \Lambda_{(\mu)_2}[g_3(r)]\}$ is continuous in the set I_2^+, and functions $g_j(r)$ satisfy the boundary conditions

$$\left(\tilde{\alpha}_{11}^0\frac{d}{dr} + \tilde{\beta}_{11}^0\right)g_1(r)\Big|_{r=R_0} = g_0,$$

$$\lim_{r\to+\infty}\left[shr\left(\frac{dg_3}{dr}V_{\upsilon,\alpha;3}^{(\mu)} - g_3\frac{dV_{\upsilon,\alpha;3}^{(\mu)}}{dr}\right)\right] = 0 \qquad (26)$$

and the conjugate conditions

$$\left[\left(\tilde{\alpha}_{j1}^{k}\frac{d}{dr}+\tilde{\beta}_{j1}^{k}\right)g_k(r)-\left(\tilde{\alpha}_{j2}^{k}\frac{d}{dr}+\tilde{\beta}_{j2}^{k}\right)g_{k+1}(r)\right]\Bigg|_{r=R_k}=\omega_{jk}, \quad (27)$$

$$j,k=1,2,$$

then the basic identity for the HIT of the HDO $M_{\upsilon,\alpha}^{(\mu)}$ is true:

$$H_{\upsilon,\alpha}^{(\mu)}\left[M_{\upsilon,\alpha}^{(\mu)}[g(r)]\right]=-\beta^2\tilde{g}(\beta)-\sum_{i=1}^{3}k_i^2\tilde{g}_i(\beta)+(-\tilde{\alpha}_{11}^{0})^{-1}V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta)\times$$
$$\times a_1^2\sigma_1 shR_0 g_0+\sum_{k=1}^{2}d_k\left[Z_{\upsilon,\alpha;12}^{(\mu),k}(\beta)\omega_{2k}-Z_{\upsilon,\alpha;22}^{(\mu),k}(\beta)\omega_{1k}\right]. \quad (28)$$

Proof. According the rule (23) we have that

$$H_{\upsilon,\alpha}^{(\mu)}\left[M_{\upsilon,\alpha}^{(\mu)}[g(r)]\right]=\int_{R_0}^{+\infty}M_{\upsilon,\alpha}^{(\mu)}[g(r)]V_{\upsilon,\alpha}^{(\mu)}(r,\beta)\sigma(r)dr=$$

$$=\int_{R_0}^{R_1}a_1^2\Lambda_{(\mu)_1}[g_1(r)]V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta)\sigma_1 shrdr+\int_{R_1}^{R_2}a_2^2B_{\upsilon,\alpha}[g_2(r)]V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta)\times \quad (29)$$
$$\times\sigma_2 r^{2\alpha+1}dr+\int_{R_2}^{+\infty}a_3^2\Lambda_{(\mu)_2}[g_3(r)]V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)\sigma_3 shrdr\,.$$

Let's integrate by parts twice in (29). We have:

$$H_{\upsilon,\alpha}^{(\mu)}\left[M_{\upsilon,\alpha}^{(\mu)}[g(r)]\right]=a_1^2\sigma_1\left[shr\left(\frac{dg_1}{dr}V_{\upsilon,\alpha;1}^{(\mu)}-g_1\frac{dV_{\upsilon,\alpha;1}^{(\mu)}}{dr}\right)\right]\Bigg|_{R_0}^{R_1}+$$
$$+\int_{R_0}^{R_1}g_1(r)\left(a_1^2\Lambda_{(\mu)_1}[V_{\upsilon,\alpha;1}^{(\mu)}]\right)\sigma_1 shrdr+a_2^2\sigma_2\times$$
$$\times\left[r^{2\alpha+1}\left(\frac{dg_2}{dr}V_{\upsilon,\alpha;2}^{(\mu)}-g_2\frac{dV_{\upsilon,\alpha;2}^{(\mu)}}{dr}\right)\right]\Bigg|_{R_1}^{R_2}+ \quad (30)$$
$$+\int_{R_1}^{R_2}g_2(r)\left(a_2^2B_{\upsilon,\alpha}[V_{\upsilon,\alpha;2}^{(\mu)}]\right)\sigma_2 r^{2\alpha+1}dr+$$
$$+a_3^2\sigma_3\left[shr\left(\frac{dg_3}{dr}V_{\upsilon,\alpha;3}^{(\mu)}-g_3\frac{dV_{\upsilon,\alpha;3}^{(\mu)}}{dr}\right)\right]\Bigg|_{R_2}^{+\infty}+\int_{R_2}^{+\infty}g_3\left(a_3^2\Lambda_{(\mu)_2}[V_{\upsilon,\alpha;3}^{(\mu)}]\right)\sigma_3 shrdr\,.$$

If $\tilde{\alpha}_{11}^{0}\neq 0$, then

$$-a_1^2\sigma_1 shR_0\left(g_1'V_{\upsilon,\alpha;1}^{(\mu)}-g_1V_{\upsilon,\alpha;1}^{(\mu)\,\prime}\right)\Big|_{r=R_0}=a_1^2\sigma_1(-\tilde{\alpha}_{11}^{0})^{-1}shR_0\times$$

$$\times\left[\left(\tilde{\alpha}_{11}^{0}\frac{dg_1}{dr}+\tilde{\beta}_{11}^{0}g_1\right)\Bigg|_{r=R_0} V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta)-(\tilde{\alpha}_{11}^{0})^{-1}\tilde{\beta}_{11}^{0}g_1(R_0)V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta)-\right.$$

$$\left.-g_1(R_0)\frac{dV_{\upsilon,\alpha;1}^{(\mu)}}{dr}\Bigg|_{r=R_0}\right]=(-\tilde{\alpha}_{11}^{0})^{-1}a_1^2\sigma_1 shR_0 g_0 V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta)+g_1(R_0)(\tilde{\alpha}_{11}^{0})^{-1}\times \tag{31}$$

$$\times a_1^2\sigma_1\left(\tilde{\alpha}_{11}^{0}\frac{dV_{\upsilon,\alpha;1}^{(\mu)}}{dr}+\tilde{\beta}_{11}^{0}V_{\upsilon,\alpha;1}^{(\mu)}\right)\Bigg|_{r=R_0}=(-\tilde{\alpha}_{11}^{0})^{-1}V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta)a_1^2\sigma_1 shR_0\cdot g_0$$

Due to the boundary condition in the $r\to+\infty$ we have that

$$\lim_{r\to+\infty}\left[shr\left(g_3'(r)V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)-g_3(r){V_{\upsilon,\alpha;3}^{(\mu)}}'(r,\beta)\right)\right]=0\,. \tag{32}$$

Let's use the basic identity (8) in the point $r=R_1$ for the case if the conjugate conditions are not homogeneous:

$$a_1^2\sigma_1 shR_1\left(g_1'V_{\upsilon,\alpha;1}^{(\mu)}-g_1{V_{\upsilon,\alpha;1}^{(\mu)}}'\right)\Bigg|_{r=R_1}-a_2^2\sigma_2R_1^{2\alpha+1}\left(g_2'V_{\upsilon,\alpha;2}^{(\mu)}-\right.$$

$$\left.-g_2{V_{\upsilon,\alpha;2}^{(\mu)}}'\right)\Bigg|_{r=R_1}=\left(a_1^2\sigma_1 shR_1\frac{c_{21,1}}{c_{11,1}}-a_2^2\sigma_2R_1^{2\alpha+1}\right)\times \tag{33}$$

$$\times\left(g_2'V_{\upsilon,\alpha;2}^{(\mu)}-g_2{V_{\upsilon,\alpha;2}^{(\mu)}}'\right)\Bigg|_{r=R_1}+a_1^2\sigma_1 shR_1 : c_{11,1}\left(Z_{\upsilon,\alpha;12}^{(\mu);1}(\beta)\omega_{21}-\right.$$

$$\left.-Z_{\upsilon,\alpha;22}^{(\mu);1}(\beta)\omega_{11}\right)=d_1\left(Z_{\upsilon,\alpha;12}^{(\mu);1}(\beta)\omega_{21}-Z_{\upsilon,\alpha;22}^{(\mu);1}(\beta)\omega_{11}\right),$$

because due to the choice of numbers σ_1 and σ_2 expression

$$a_1^2\sigma_1 shR_1\frac{c_{21,1}}{c_{11,1}}-a_2^2\sigma_2R_1^{2\alpha+1}=\frac{c_{11,2}}{c_{21,2}}\frac{R_1^{2\alpha+1}}{R_2^{2\alpha+1}}shR_2-\frac{c_{11,2}}{c_{21,2}}\frac{R_1^{2\alpha+1}}{R_2^{2\alpha+1}}shR_2=0\,.$$

Similarly we find in the conjugate point $r=R_2$ that

$$a_2^2\sigma_2R_2^{2\alpha+1}\left(g_2'V_{\upsilon,\alpha;2}^{(\mu)}-g_2{V_{\upsilon,\alpha;2}^{(\mu)}}'\right)\Bigg|_{r=R_2}-a_3^2\sigma_3\left(g_3'V_{\upsilon,\alpha;3}^{(\mu)}-\right.$$

$$\left.-g_3{V_{\upsilon,\alpha;3}^{(\mu)}}'\right)\Bigg|_{r=R_2}\cdot shR_2=\left(a_2^2\sigma_2R_2^{2\alpha+1}\frac{c_{21,2}}{c_{11,2}}-a_3^2\sigma_3 shR_2\right)\times \tag{34}$$

$$\times\left(g_3'(R_2)V_{\upsilon,\alpha;3}^{(\mu)}(R_2,\beta)-g_3(R_2){V_{\upsilon,\alpha;3}^{(\mu)}}'(R_2,\beta)\right)+a_2^2\sigma_2R_2^{2\alpha+1}\cdot c_{11,2}^{-1}\times$$

$$\times\left(Z_{\upsilon,\alpha;12}^{(\mu);2}(\beta)\omega_{22}-Z_{\upsilon,\alpha;22}^{(\mu);2}(\beta)\omega_{12}\right)=d_2\left(Z_{\upsilon,\alpha;12}^{(\mu);2}(\beta)\omega_{22}-Z_{\upsilon,\alpha;22}^{(\mu);2}(\beta)\omega_{12}\right)$$

because due to the choice of numbers σ_2 and σ_3 expression

$$a_2^2\sigma_2 R_2^{2\alpha+1}\frac{c_{21,2}}{c_{11,2}} - a_3^2\sigma_3 shR_2 = \frac{c_{11,2}}{c_{21,2}} shR_2 \frac{c_{21,2}}{c_{11,2}} - shR_2 = shR_2(1-1) = 0\ .$$

From the differential identities

$$[a_1^2\Lambda_{(\mu)_1} + (\beta^2 + k_1^2)]V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta) = 0\ ,$$

$$[a_2^2 B_{\upsilon,\alpha} + (\beta^2 + k_2^2)]V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta) = 0\ ,$$

$$[a_3^2\Lambda_{(\mu)_2} + (\beta^2 + k_3^2)]V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta) = 0$$

we find the differential equality:

$$\begin{aligned} a_1^2\Lambda_{(\mu)_1}[V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta)] &= -(\beta^2 + k_1^2)V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta), \\ a_2^2 B_{\upsilon,\alpha}[V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta)] &= -(\beta^2 + k_2^2)V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta), \\ a_3^2\Lambda_{(\mu)_2}[V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)] &= -(\beta^2 + k_3^2)V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta). \end{aligned} \qquad (35)$$

Let's substitute the obtained identical equality (31)–(35) to the (30). We get:

$$\begin{aligned} H_{\upsilon,\alpha}^{(\mu)}\left[M_{\upsilon,\alpha}^{(\mu)}[g(r)]\right] &= (-\alpha_{11}^0)^{-1}V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta)a_1^2\sigma_1 shR_0 g_0 + \\ &+\sum_{k=1}^{2} d_k\left[Z_{\upsilon,\alpha;12}^{(\mu),k}(\beta)\omega_{2k} - Z_{\upsilon,\alpha;22}^{(\mu),k}(\beta)\omega_{1k}\right] - (\beta^2 + k_1^2)\tilde{g}_1(\beta) - \\ &-(\beta^2 + k_2^2)\tilde{g}_2(\beta) - (\beta^2 + k_3^2)\tilde{g}_3(\beta)\ . \end{aligned} \qquad (36)$$

Since the $[-\beta^2(\tilde{g}_1(\beta) + \tilde{g}_2(\beta) + \tilde{g}_3(\beta))] = -\beta^2\tilde{g}(\beta)$, and $[-k_1^2\tilde{g}_1(\beta) - -k_2^2\tilde{g}_2(\beta) - k_3^2\tilde{g}_3(\beta)] = -\sum_{i=1}^{3} k_i^2\tilde{g}_i(\beta)$, then equality (36) coincides with equality (28). The theorem is proved.

Rules (23), (24) and (28) constitute the mathematical apparatus for solving parabolic problem (1)–(4) by the famous logical scheme [7].

Let's write the system (1) and initial conditions (2) in matrix form:

$$\begin{bmatrix} \left(\dfrac{\partial}{\partial t} + \gamma_1^2 - a_1^2\Lambda_{(\mu)_1}\right)u_1(t,r) \\ \left(\dfrac{\partial}{\partial t} + \gamma_2^2 - a_2^2 B_{\upsilon,\alpha}\right)u_2(t,r) \\ \left(\dfrac{\partial}{\partial t} + \gamma_3^2 - a_3^2\Lambda_{(\mu)_2}\right)u_3(t,r) \end{bmatrix} = \begin{bmatrix} f_1(t,r) \\ f_2(t,r) \\ f_3(t,r) \end{bmatrix},\quad \left.\begin{bmatrix} u_1(t,r) \\ u_2(t,r) \\ u_3(t,r) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} g_1(r) \\ g_2(r) \\ g_3(r) \end{bmatrix}. \qquad (37)$$

The integral operator $H_{\upsilon,\alpha}^{(\mu)}$ is represented as an operator matrix- row due to the rule (23):

$$H_{\upsilon,\alpha}^{(\mu)}[\cdots] = \left[\int_{R_0}^{R_1} \cdots V_{\upsilon,\alpha;1}^{(\mu)}(r,\beta)\sigma_1 shr dr \quad \int_{R_1}^{R_2} \cdots V_{\upsilon,\alpha;2}^{(\mu)}(r,\beta)\sigma_2 r^{2\alpha+1} dr \right.$$
$$\left. \int_{R_2}^{+\infty} \cdots V_{\upsilon,\alpha;3}^{(\mu)}(r,\beta)\sigma_3 shr dr \right]. \tag{38}$$

Let's apply the operator matrix-row (38) to the problem (37) according to matrices multiplication rule. As a result of main identity (28) we get a Cauchy problem [13]:

$$\left[\frac{d}{dt} + (\beta^2 + \gamma^2)\right]\tilde{u}(t,\beta) =$$
$$= \tilde{f}(t,\beta) + (-\tilde{\alpha}_{11}^0)^{-1} V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta) a_1^2 \sigma_1 sh R_0 g_0(t) + \tag{39}$$
$$+\sum_{k=1}^{2} d_k \left[Z_{\upsilon,\alpha;12}^{(\mu),k}(\beta)\omega_{2k}(t) - Z_{\upsilon,\alpha;22}^{(\mu),k}(\beta)\omega_{1k}(t) \right],$$
$$\gamma^2 = \max\{\gamma_1^2; \gamma_2^2; \gamma_3^2\}$$

$$\tilde{u}(t,\beta)\big|_{t=0} \equiv \sum_{i=1}^{3} \tilde{u}_i(t,\beta)\big|_{t=0} = \tilde{g}(\beta) \equiv \sum_{i=1}^{3} \tilde{g}_i(\beta)\,. \tag{40}$$

Remark 3. We obtain equality (39) under the assumption that initial conditions satisfy equalities: $\delta_{11}^0 g_1'(R_0) + \gamma_{11}^0 g_1(R_0) \equiv \psi_0 = 0$, $\psi_{jk} \equiv$ $\equiv [\delta_{j1}^k g_k'(R_k) + \gamma_{j1}^k g_k(R_k)] - [\delta_{j2}^k g_{k+1}'(R_k) + \gamma_{j2}^k g_{k+1}(R_k)] = 0$. Otherwise, we go to the new initial conditions $g_j(r) = g_j^*(r) + a_j r + b_j$, де $j = \overline{1,3}$, $a_3 = 0$ and choose values a_1, a_2 та b_1, b_2, b_3 from algebraic system of equations:

$$\left(\delta_{11}^0 + \gamma_{11}^0 R_0\right) a_1 + \gamma_{11}^0 b_1 = \psi_0, \tag{41}$$
$$[(\delta_{j1}^k + \gamma_{j1}^k R_k) a_k + \gamma_{j1}^k b_k] - [(\delta_{j2}^k + \gamma_{j2}^k R_k) a_{k+1} + \gamma_{j2}^k b_{k+1}] = \psi_{jk}\,;\ a_3 = 0\,.$$

With assumptions on the coefficients the algebraic system (41) always has a unique solution.

It is directly verify that the solution of the Cauchy problem (39), (40) is a function

$$\tilde{u}(t,\beta) = \int_0^t e^{-(\beta^2+\gamma^2)(t-\tau)} [\tilde{f}(\tau,\beta) + \delta_+(\tau)\tilde{g}(\beta)] d\tau + \int_0^t e^{-(\beta^2+\gamma^2)(t-\tau)} \times$$
$$\times g_0(\tau) d\tau (-\tilde{\alpha}_{11}^0)^{-1} V_{\upsilon,\alpha;1}^{(\mu)}(R_0,\beta) a_1^2 \sigma_1 sh R_0 + \sum_{k=1}^{2} d_k \left[\int_0^t e^{-(\beta^2+\gamma^2)(t-\tau)} \times \right. \tag{42}$$

$$\times\omega_{2k}(\tau)d\tau-\int_0^t e^{-(\beta^2+\gamma^2)(t-\tau)}\omega_{1k}(\tau)d\tau\Bigg],$$

where $\delta_+(\tau)$ — Dirac delta-function focused in the point +0 [14].

Integral operator $H^{-(\mu)}_{\upsilon,\alpha}$ due to the rule (24), as inverse to (38), we represent as the operator matrix-column:

$$H^{-(\mu)}_{\upsilon,\alpha}=\begin{bmatrix}\frac{2}{\pi}\int_0^\infty\cdots V^{(\mu)}_{\upsilon,\alpha;1}(r,\beta)\Omega^{(\mu)}_{\upsilon,\alpha}(\beta)d\beta\\ \frac{2}{\pi}\int_0^\infty\cdots V^{(\mu)}_{\upsilon,\alpha;2}(r,\beta)\Omega^{(\mu)}_{\upsilon,\alpha}(\beta)d\beta\\ \frac{2}{\pi}\int_0^\infty\cdots V^{(\mu)}_{\upsilon,\alpha;3}(r,\beta)\Omega^{(\mu)}_{\upsilon,\alpha}(\beta)d\beta\end{bmatrix}. \tag{43}$$

Let's apply operator matrix-column (43) to matrix element $[\tilde{u}(t,\beta)]$ due to matrices multiplication rule, where the function $\tilde{u}(t,\beta)$ is defined by formula (42). As a result of elementary transformations, we get the integral representation of the only exact analytical solution of parabolic problem (1)–(4):

$$u_j(t,r)=\int_0^t W^{(\mu)}_{\upsilon,\alpha;1j}(t-\tau,r)g_0(\tau)d\tau+\sum_{k=1}^{2}dk\Bigg[\int_0^t R^{(\mu);k,j}_{\upsilon,\alpha;12}(t-\tau,r)\omega_{2k}(\tau)d\tau-$$

$$-\int_0^t R^{(\mu);k,j}_{\upsilon,\alpha;22}(t-\tau,r)\omega_{1k}(\tau)d\tau\Bigg]+\int_0^t\int_{R_0}^{R_1}H^{(\mu)}_{\upsilon,\alpha;j1}(t-\tau,r,\rho)[f_1(\tau,\rho)+$$

$$+\delta_+(\tau)g_1(\rho)]\sigma_1 sh\rho d\rho d\tau+\int_0^t\int_{R_1}^{R_2}H^{(\mu)}_{\upsilon,\alpha;j2}(t-\tau,r,\rho)[f_2(\tau,\rho)+\delta_+(\tau)g_2(\rho)]\times \tag{44}$$

$$\times\sigma_2\rho^{2\alpha+1}d\rho d\tau+\int_0^t\int_{R_2}^{+\infty}H^{(\mu)}_{\upsilon,\alpha;j3}(t-\tau,r,\rho)[f_3(\tau,\rho)+\delta_+(\tau)g_3(\rho)]\times$$

$$\times\sigma_3 sh\rho d\rho d\tau,\quad j=\overline{1,3}.$$

In equalities (44) there are principal solutions of problem:

1) Green's functions generated by boundary condition at point $r=R_0$

$$W^{(\mu)}_{\upsilon,\alpha;1j}(t,r)=\frac{2}{\pi}\int_0^{+\infty}(-\tilde{\alpha}^0_{11})^{-1}V^{(\mu)}_{\upsilon,\alpha;1}(R_0,\beta)V^{(\mu)}_{\upsilon,\alpha;j}(r,\beta)\Omega^{(\mu)}_{\upsilon,\alpha}(\beta)\times$$

$$\times e^{-(\beta^2+\gamma^2)t}d\beta a_1^2\sigma_1 shR_0;$$

2) Green's functions generated by inhomogeneity of the conjugate conditions

$$R_{\upsilon,\alpha;i2}^{(\mu);k,j}(t,r)=\frac{2}{\pi}\int_0^{+\infty} Z_{\upsilon,\alpha;i2}^{(\mu);k}(\beta)V_{\upsilon,\alpha;j}^{(\mu)}(r,\beta)\Omega_{\upsilon,\alpha}^{(\mu)}(\beta)\times$$
$$\times e^{-(\beta^2+\gamma^2)t}d\beta;\quad i,k=1,2;\ j=\overline{1,3};$$

3) the influence functions generated by the inhomogeneity of system (of initial conditions)

$$H_{\upsilon,\alpha;jk}^{(\mu)}(t,r,\rho)=\frac{2}{\pi}\int_0^{+\infty} e^{-(\beta^2+\gamma^2)t}V_{\upsilon,\alpha;j}^{(\mu)}(r,\beta)V_{\upsilon,\alpha;k}^{(\mu)}(\rho,\beta)\times$$
$$\times\Omega_{\upsilon,\alpha}^{(\mu)}(\beta)d\beta;\quad j,k=\overline{1,3}.$$

We get the following theorem as the summary of the written above.

Theorem 3. Let the next conditions be true:

1) functions $f_j(t,r)$, $g_0(t)$ and $\omega_{jk}(t)$ are originals by Laplace relative to variable t [15];
2) functions $f_j(t,r)$ and $g_j(r)$ satisfy the conjugate conditions;
3) functions $f(t,r)=\{f_1(t,r);f_2(t,r);f_3(t,r)\}$ and $g(r)=\{g_1(r);g_2(r);g_3(r)\}$ are bounded, continuous, absolutely summable with the weight function $\sigma(r)$ and have the bounded variation in the set I_2^+;
4) function $F(t,r)=\left\{\frac{\partial}{\partial r}\Lambda_{(\mu)_1}[f_1(t,r)];\frac{\partial}{\partial r}B_{\upsilon,\alpha}[f_2(t,r)];\frac{\partial}{\partial r}\Lambda_{(\mu)_2}[f_3(t,r)]\right\}$ is continuously differentiable by t and continuous by r in the set D_2^+.

Then in the class of functions $u(t,r)=\{u_1(t,r);u_2(t,r);u_3(t,r)\}$, which are continuously differentiable by variable t and continuously differentiable by variable r in the set D_2^+ and satisfy conditions 1), 3), parabolic mixed problem (1)–(4) has unique bounded solution, which is determined by the formula (44).

Remark 4. If $\gamma^2=\gamma_1^2>0$, then $k_1^2=0$, $k_2^2=\gamma_1^2-\gamma_2^2\ge 0$, $k_3^2=\gamma_1^2-\gamma_3^2\ge 0$; if $\gamma^2=\gamma_2^2>0$, then $k_1^2=\gamma_2^2-\gamma_1^2\ge 0$, $k_2^2=0$, $k_3^2=\gamma_2^2-\gamma_3^2\ge 0$; if $\gamma^2=\gamma_3^2>0$, then $k_1^2=\gamma_3^2-\gamma_1^2\ge 0$, $k_2^2=\gamma_3^2-\gamma_2^2\ge 0$, $k_3^2=0$.

Conclusions. By means of method of hybrid integral transform of Legendre-Bessel-Legendre type with the spectral parameter in conjunction with the method of principal solutions (influence functions and Green functions) exact analytical solution of the mixed problem for a system of evolution equations of parabolic type, modeling by hybrid differential Legendre-Bessel-Legendre operator in the piece-homogeneous polar axis $r \geq R_0 > 0$ with soft limits is obtained.

The obtained solutions are of algorithmic character, continuously depend on the parameters and data problems and can be used in further theoretical research and in practical engineering calculations (involving methods of computer algebra) of real evolutionary processes in piece-homogeneous environments which modeled by parabolic boundary-value problems (heat conduction processes, diffusion).

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Методом гібридного інтегрального перетворення типу Лежандра-Бесселя-Лежандра зі спектральним параметром одержано інтегральне зображення точного аналітичного розв'язку мішаної задачі для системи рівнянь параболічного типу на кусково-однорідній полярній осі $r \geq R_0 > 0$ з м'якими межами. Моделювання еволюційного процесу здійснено методом гібридного диференціального оператора Лежандра-Бесселя-Лежандра.

Ключові слова: *параболічне рівняння, початкові та крайові умови, умови спряження, диференціальний оператор, гібридне інтегральне перетворення, функції Гріна, функції впливу.*

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В. Р. Кукурба, аспірант

Національний університет «Львівська політехніка», м. Львів

СТОХАСТИЧНА ОПТИМІЗАЦІЯ З НАПІВМАРКОВСЬКИМИ ПЕРЕКЛЮЧЕННЯМИ ТА ІМПУЛЬСНИМИ ЗБУРЕННЯМИ

Розглядається неперервна процедура стохастичної оптимізації з імпульсними збуреннями в напівмарковськовському середовищі. Для функція регресії, що залежить від рівномірно ергодичного напівмарковського процесу, встановлено достатні умови збіжності через властивості компенсуючого оператора розширеного процесу марковського відновлення процедури та його асимптотичне представлення на збуреній функції Ляпунова.

Ключові слова: *стохастична оптимізація, напівмарковський процес, компенсуючий оператор, імпульсні збурення.*

Вступ. Розглядається задача оптимізації у випадковому середовищі, що описується напівмарковським процесом, і полягає у встановленні умов збіжності процедури стохастичної оптимізації (ПСО). Широкий спектр застосування оптимізаційних процедур при обробці та аналізі експерементальних даних, оптимізації складних систем [1] та задачах розпізнавання образів підкреслює актуальність нових застосування оптимізаційних процедур.

Оптимізаційна процедура Кіфера-Вольфовиця [2] для функції регресії $C(u)$, полягає у розв'язку рівняння регресії $C'(u) = 0$, тобто у

