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ESTIMATION OF THE BEST APPROXIMATIONS FOR THE GENERALIZED DERIVATIVE IN BANACH SPACES

The main task of the theory of approximation is to establish the properties of the approximation characteristics of this function on the basis of its investigated properties. Functions with the same properties are grouped into classes, and then the facts established for a particular class apply to each of its representatives. This makes it possible to formulate new problems, in particular mathematical modeling problems for whole classes of functions that describe the studied processes.

If the statements allow on the basis of information about the generalized derivative of the element *f* to draw a conclusion about the rate of approach to zero of the sequence of the best approximations of this element by polynomials of degree n, then in the theory of approximations they are called direct theorems.

In the given article the inverse theorem is considered — per properties of sequence of the best approximations we draw a conclusion about properties of an element *f* of some Banach space *B* and its generalized derivatives. That is, according to a given sequence of the best approximations of the vector *f* by polynomials of degree *n* establish its differential characteristics.

The first inverse theorems were considered at the beginning of the last century by S. N. Bernstein. The main point of their proof is the inequalities between the norms of polynomials and their derivatives. Such inequalities are called Bernstein inequalities. As a partial case, they can be obtained from the theorem considered in the article.

Key words: *the best approximations, generalized derivatives, inverse theorems, Bernstein inequalities, differential characteristics, Banach space.*

Introduction. The main task of the theory of approximation is to establish the properties of the approximation characteristics of this function on the basis of its investigated properties. Functions with the same properties are grouped into classes, and then the facts established for a particular class apply to each of its representatives. This makes it possible to formulate new problems, in particular mathematical modeling problems for whole classes of functions that describe the studied processes.

Consider the problem of the properties of the best approximations for a generalized derivative in Banach spaces.

Problem statement. Suppose that in Banach space B a complete minimal system $\{\varphi_m\}_{m=1}^{\infty}$ is given, and $\left\{\varphi_{m=1}^{*}\right\}_{m=1}^{\infty}$ is its conjugate system belonging to B^*, $\{\lambda_m\}_{m=1}^{\infty}$ — a sequence of complex numbers.

Definition. If for the element $f \in B$ the sum of the series $\sum_{m=1}^{\infty} \lambda_m \left(f, \varphi_m^* \right) \varphi_m$ is some element $g \in B$, then the vector g is called the derivative of the vector f and is denoted by $\partial_{\varphi}^{\lambda}$, namely

$$\partial_{\varphi}^{\lambda} f = \sum_{m=1}^{\infty} \lambda_m \left(f, \varphi_m^* \right) \varphi_m. \tag{1}$$

The subset of all vectors $f \in B$ having $\partial_{\varphi}^{\lambda}$ — derivatives will be denoted by $V\ (\partial_{\varphi}^{\lambda})$.

Vector

$$T_n(\varphi) = \sum_{m=1}^{n} c_m \varphi_m\ , \tag{2}$$

where c_m are arbitrary complex numbers, let's call a polynomial of degree n according to the system $\{\varphi_m\}_{m=1}^{\infty}$. Note that due to the minimality of the system $\{\varphi_m\}_{m=1}^{\infty}$, the coefficients c_m in (2) are uniquely determined by the vector $T_n(\varphi)$ and $c_l = \left(T_n(\varphi), \varphi_l^* \right)$.

We enter values

$$E_n(f, \varphi_m) = \inf_{T_n(\varphi)} \left\| f - T_n(\varphi) \right\|, n = 1, 2, ..., \tag{3}$$

which are called the best approximations of the vector f by polynomials of degree n over the system $\{\varphi_m\}_{m=1}^{\infty}$. We will consider $E_0(f) = \|f\|$.

It is obvious that $E_n(f) \geq E_{n+1}(f),\ n = 0, 1, ...$ and due to the completeness of the system $\{\varphi_m\}_{m=1}^{\infty}$ is a valid statement $E_n(f) \to 0$ if $n \to \infty$, ie $E_n(f) \downarrow 0$, $n \to \infty$.

If the statements allow on the basis of information about the generalized $\partial_{\varphi}^{\lambda}$ -derivative of the element f to draw a conclusion about the rate of approach to zero of the sequence of the best approximations of this element by polynomials of degree n, then in the theory of approximations they are called direct theorems.

In the given article the inverse theorem is considered — per properties of sequence of the best approximations $E_n(f)$ we draw a conclusion about properties of an element $f \in B$ and its generalized $\partial_{\varphi}^{\lambda}$ -derivatives. That is, according to a given sequence $E_n(f)$ of the best approximations of the vector f by polynomials of degree n according to the system $\{\varphi_m\}_{m=1}^{\infty}$, establish its differential characteristics.

The first inverse theorems were considered at the beginning of the last century by S. N. Bernstein [2]. The main point of their proof is the inequalities between the norms of polynomials and their derivatives. Such inequalities are called Bernstein inequalities. As a partial case, they can be obtained from the theorem considered in the article.

Due to the linear independence of vectors φ_m, steels exist

$$\mu_n\left(\partial_{\varphi}^{\lambda}\right) = \sup_{\|T_n(\varphi_m)\|=1} \left\|\partial_{\varphi}^{\lambda}\, T_n(\varphi_m)\right\|,\ n = 1, 2, \ldots, \tag{4}$$

which we will call the Szego constants for the $\partial_{\varphi}^{\lambda}$ -derivative (see [1, 3, 7]).

Obviously $\mu_{n+1}\left(\partial_{\varphi}^{\lambda}\right) \ge \mu_n\left(\partial_{\varphi}^{\lambda}\right) > 0$.

In the accepted designations the following statement takes place.

Theorem. Suppose that for some increasing sequence of natural numbers $\{n_l\}_{l=1}^{\infty}$ the series

$$\sum_{l=1}^{\infty} \mu_{n_{l+1}}\left(\partial_{\varphi}^{\lambda}\right) E_{n_l}(f,\varphi) \tag{5}$$

convergents. Then $f \in V(\partial_{\varphi}^{\lambda})$ and

$$E_{n_j}\left(\partial_{\varphi}^{\lambda} f,\varphi\right) \le 2\sum_{l=j}^{\infty} \mu_{n_{l+1}}\left(\partial_{\varphi}^{\lambda}\right) E_{n_l}(f,\varphi). \tag{6}$$

Proof. For an arbitrary $\varepsilon > 0$ we choose the polynomials $T_n(\varphi)$ for which

$$\left\| f - T_n(\varphi)\right\| \le (1+\varepsilon_n) E_n(f,\varphi) \tag{7}$$

and we show that the sequence $\partial_{\varphi}^{\lambda} T_{n_l}(\varphi)$ is fundamental.

Let $s > j$, then

$$\partial_{\varphi}^{\lambda} T_{n_s}(\varphi) - \partial_{\varphi}^{\lambda} T_{n_j}(\varphi) = \sum_{l=j}^{s-1} \partial_{\varphi}^{\lambda} \left(T_{n_{l+1}}(\varphi) - T_{n_l}(\varphi) \right), \qquad (8)$$

where $T_{n_{l+1}}(\varphi) - T_{n_l}(\varphi)$ is a polynomial of degree n according to the system $\{\varphi_m\}_{m=1}^{\infty}$. Then from (4) and (7) it follows that:

$$\left\| \partial_{\varphi}^{\lambda} \left(T_{n_{l+1}}(\varphi) - T_{n_l}(\varphi) \right) \right\| \le \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) \cdot \left\| T_{n_{l+1}}(\varphi) - T_{n_l}(\varphi) \right\| \le$$

$$\le \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) \cdot \left(\left\| f - T_{n_{l+1}}(\varphi) \right\| + \left\| f - T_{n_l}(\varphi) \right\| \right) \le$$

$$\le \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) (1+\varepsilon) \left(E_{n_{l+1}}(f,\varphi) + E_{n_l}(f,\varphi) \right) \le$$

$$\le 2 \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) (1+\varepsilon) E_{n_l}(f,\varphi).$$

Substituting the obtained inequality in (8), we have an estimate

$$\left\| \partial_{\varphi}^{\lambda} T_{n_s}(\varphi) - \partial_{\varphi}^{\lambda} T_{n_j}(\varphi) \right\| \le (1+\varepsilon) \cdot 2 \sum_{l=j}^{s-1} \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) E_{n_l}(f,\varphi). \qquad (9)$$

Given the convergence of the series (5) and relation (9), we conclude that the sequence $\partial_{\varphi}^{\lambda} T_{n_j}(\varphi)$ is fundamental. Therefore, there is an element $g \in B$, for which $\left\| g - \partial_{\varphi}^{\lambda} T_{n_j}(\varphi) \right\| \to 0$, and which by definition is called a $\partial_{\varphi}^{\lambda}$-derivative of the vector f.

Passaging to the limit in the inequality (9), when $s \to \infty$, we obtain:

$$\left\| \partial_{\varphi}^{\lambda} f - \partial_{\varphi}^{\lambda} T_{n_j}(\varphi) \right\| \le (1+\varepsilon) \cdot 2 \sum_{l=j}^{\infty} \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) E_{n_l}(f,\varphi), \qquad (10)$$

Since $\partial_{\varphi}^{\lambda} T_{n_j}(\varphi)$ is a polynomial of degree n_j according to the system $\{\varphi_m\}_{m=1}^{\infty}$, we have:

$$E_{n_j}\left(\partial_{\varphi}^{\lambda} f, \varphi \right) \le \left\| \partial_{\varphi}^{\lambda} f - \partial_{\varphi}^{\lambda} T_{n_j}(\varphi) \right\| \le 2 \cdot (1+\varepsilon) \sum_{l=j}^{\infty} \mu_{n_{l+1}} \left(\partial_{\varphi}^{\lambda} \right) E_{n_l}(f,\varphi).$$

Taking into account the arbitrariness of ε, we obtain the estimate (6).

Theorem proved.

If

$$B = L_P, \; L_P = L_P[0; 2\pi], \; 1 \le p < \infty, \; \|f\|_p = \|f\|_{L_p} = \left(\int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p},$$

$$\varphi_m(x) = e^{imx},\ m \in Z\ ,\ \lambda_m = e^{i\frac{\pi}{2}\beta signm}\psi^{-1}(|m|),$$

here $\psi(|m|)$ — a sequence of nonnegative numbers, β is some real number, then the series (1) coincides with the Fourier series of the $(\psi;\beta)$ - derivative of the function f, introduced by Stepanets [4]. In this case, the set $V\ (\partial_\varphi^\lambda)$ becomes a class of functions L_β^ψ, and the vector $T_n(\varphi)$ (2) becomes a tr igonometric polynomial

$$T_n(x) = \sum_{m=-n}^{n} c_m e^{imx}.$$

$E_n(f,\varphi_m)$ for this case will be the best approximation of the function f by trigonometric polynomials:

$$E_n(f) = \inf_{T_{n-1}\in\mathcal{T}_{2n-1}} f - T_{n-1}(\cdot)_P\ ,\ 1 \le p < \infty,$$

here $\mathcal{T}_{2n-1}$ — the set of trigonometric polynomials of degree not greater than $n-1$. From condition (4) it follows that

$$\partial_\varphi^\lambda T_n(\varphi_m) \le \sup_{T_n(\varphi_m)=1} \partial_\varphi^\lambda T_n(\varphi_m) = \mu_n\left(\partial_\varphi^\lambda\right)T_n(\varphi)) .$$

In the above notations $\partial_\varphi^\lambda T_n(\varphi_m)$ coincides with the polynomials $T_n(\cdot)_\beta^\psi$ and for $\mu = O(1)\psi^{-1}(n)$, where $O(1)$ is a quantity uniformly bounded with respect to n, we have the inequality:

$$\| T_n(n)_\beta^\psi \|_p \le O(1)\psi^{-1}(n)T_n(\cdot)_P \tag{11}$$

If inequality (11) holds for an arbitrary trigonometric polynomial $T_n(\cdot)$ of degree n, then we say, that the pair $(\psi;\ \beta)$ belongs to the set B_p ([5, p. 115]) If the space L_p is replaced by C, $\psi(k) = k^{-1}, k = 1, 2, \dots$, and $\beta = 1$, then relation (11) becomes a known inequality of S. N. Bernstein (in it the value of $O(1)$ can be replaced by 1).

And if the pair $(\psi;\ \beta)$ belongs to the set B_p, we can conclude that from the theorem, proved above, the statement 1) of Theorem 9.1 ([5, p. 120]) follows. This is the so-called inverse theorem, which establishes the differential characteristics of the function f with C or L_p, $1 \le p < \infty$ for a given sequence of best approximations of this function. Also from the theorem follows the corollary proved as a separate statement in [6].

Corollary. Let f belong to $L_P[0;2\pi]$, $1 \le p < \infty$ and $E_n(f)_p$ be the best approximation by trigonometric polynomials of degree not greater than $n-1$. Then, if the pair $(\psi;\ \beta) \in B_p$ and the series $\sum_{m=1}^{\infty} E_m(f)_p (\psi(m))^{-1}$ convergents, then in the function f there exists a $(\psi;\ \beta)$-derivative f_β^ψ, belonging to L_P and for which there is an inequality

$$E_n(f_\beta^\psi)_p \le K \sum_{m=n}^{\infty} E_m(f)_p (\psi(m))^{-1},\ n \in N,$$

here K is a value that may depend on $\psi(\cdot)$.

Conclusions. In the given article the inverse theorem is proved — per properties of sequence of the best approximations $E_n(f)$ we concluded about properties of an element $f \in B$ and its generalized ∂_φ^λ-derivatives. That is, according to a given sequence $E_n(f)$ of the best approximations of the vector f by polynomials of degree n according to the system $\{\varphi_m\}_{m=1}^{\infty}$, established its difference — differential characteristics. For separate values of the considered parameters, some results for $(\psi;\ \beta)$-differentiated (according to Stepanets) functions follow from the theorem, as well as the known Bernstein inequality follows.

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ОЦІНКА НАЙКРАЩИХ НАБЛИЖЕНЬ ДЛЯ УЗАГАЛЬНЕНОЇ ПОХІДНОЇ В БАНАХОВИХ ПРОСТОРАХ

Основна задача теорії наближень полягає в тому, щоб, грунтуючись на досліджуваних властивостях даної функції, встановити властивості її апроксимаційних характеристик.

Функції з однаковими властивостями об’єднуються в класи, і тоді факти, встановлені для певного класу, відносяться і до кожного його представника. При цьому з'являється можливість формулювати нові задачі, зокрема, задачі математичного моделювання вже для цілих класів функцій, які описують досліджувані процеси.

Якщо твердження дають можливість зробити висновок про швидкість прямування до нуля послідовності найкращих наближень елемента f поліномами степеня n за інформацією про узагальнену похідну цього елемента, то їх в теорії наближень прийнято називати прямими теоремами.

У статті розглядається обернена теорема — за властивостями послідовності найкращих наближень робимо висновок про властивості самого елемента f деякого банахового простору B і його узагальнених похідних, тобто за заданою послідовністю найкращих наближень вектора f поліномами степеня n встановлюються його диференціально-різницеві характеристики.

Перші обернені теореми були розглянуті ще на початку минулого століття С. Н. Бернштейном. Основним моментом їх доведення є нерівності між нормами поліномів і їх похідних. Такі нерівності називаються нерівностями Бернштейна. Як частковий випадок, вони можуть бути отримані з теореми, розглянутої в статті.

Ключові слова: *найкращі наближення, узагальнені похідні, обернені теореми, нерівності Бернштейна, диференціальні характеристики, банахів простір.*

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