The problem of convex approximation is that of estimating the degree of approximation of a convex function by convex polynomials. The problem of convex approximation is consider in [8-10]. In [8] is consider r is natural and r not equal one. In [9] is consider r is real and r more two. It was proved that for convex approximation estimates of the form (1) are fails for r is real and r more two. In [10] the question of approximation of function of Sobolev`s space and convex by algebraic convex polynomial is consider, if the index of the Sobolev space is in the interval from three to four. It is proved that the estimate that generalizes (1) is false This paper investigates the issue of approximation of convex functions from the Sobolev space by convex algebraic polynomials for a real index of the Sobolev space from the interval from two to three. Similarly to the paper [10], a counterexample is built, which shows that the estimate that generalizes the estimate (1) is false. This paper is the generalization of results papers [9] and [11]. The main result is the analog of the theorem 2.3 in [11].

Key words: *approximation of function, Sobolev space, algebraic polynomial, convex function.*

Отримано: 12.10.2022

UDC 517.5
DOI: 10.32626/2308-5878.2022-23.106-117

V. A. Sorych, Cand. of Phis. and Mathem. Sciences,
N. M. Sorych, Cand. of Phis. and Mathem. Sciences

Kamianets-Podilskyi National Ivan Ohiienko University,
Kamianets-Podilskyi

EXACT CONSTANTS OF THE BEST ONE SIDED APPROXIMATIONS OF THE SUM ANALYTIC FUNCTIONS FROM DIFFERENT CLASSES

The task of obtaining the exact values of the best approximations by trigonometric polynomials of continuous or summable functions originates from the works of P. L. Chebyshev, who, back in the 1950s, posed the problem of finding the polynomial, that deviates the least from a given continuous function. Subsequently this direction in the theory of approximation got further development thanks to the works of K. Weierstrass, D. Jackson, S. N. Bernstein, Valle-Poussin and others. On this time there is an increase attention to problems of one-sided approximation of individual functions and their classes in the metric space L. Problems of this content arise up in number theory, coding theory, and other areas of mathematics. The first results of this direction were obtained in the 1880th by A. A. Markov and T. Y. Stieltijes. In the future, these studies were continued in the works of J. Karamata (1930), G. Freud and T. Hanelius (mid-20th century).

General issues related to the problem of the best approximation of classes of functions by trigonometric polynomials: the existence of the best approximation polynomial, its characteristic properties, one sided approximations are detailed in many works, in particular, for example, in the book of M. P. Korneychuk [1], the works of T. Ganelius [4] , V. G. Doronin, A. A. Ligun [5].

The exact constants of the best one sided approximation of the sum of the majorant functions of the classes that allow analytical extension into a strip of fixed width and of functions harmonic in a circle of radius 1 have been found in this work.

Key words: *harmonic in the circle functions, analytic in the strip functions, the best joint one sided approximation.*

This article is devoted to solving one extreme problems started in the works of T Ganelius, in particular the calculation of the exact value of the best joint one sided approximation in integral metric of the sum of functions generated by Poisson kernels and analytic functions on the real axis that allow regular continuation into the strip.

Formulation of the problem. Let L_∞ — space of 2π-periodic measurable significantly limited functions with the norm $f_{L_\infty} = \| f \|_\infty = ess\ \sup \left| f(x) \right|$, C — space of continuous on real axis 2π-periodic functions $f(g)$ with the norm $f_C = \max\limits_x \left| f(x) \right|$, L — set of 2π-periodic summons on $(0;2\pi)$ functions $f(g)$ with the norm $\| f \|_L = \| f \|_1 = \int\limits_0^{2\pi} \left| f(x) \right| dx$.

Trough $\Gamma_\infty^\rho \left(\Gamma_1^\rho \right), (0 < \rho < 1)$, denote the classes of continuous 2π-periodic function served as $f(x) = U(r, x)$, where is function $U(r, x)$ is harmonic in the circle $\mathrm{r} < 1$ and satisfies inequality $\|U(r,g)\|_\infty \le 1 \left(\|U(r,g)\|_1 \le 1 \right)$ at $0 \le r < 1$. It is know, that (see, for example, [1, p.186]) classes of functions Γ_∞^ρ (Γ_1^ρ) are sets of functions represented as subsequent convolutions

$$f(x) = \frac{1}{\pi} \int\limits_0^{2\pi} \chi_\rho (x - t) \varphi(t) dt, \tag{1}$$

where $\chi_\rho(x)$ are Poisson kernels:

$$\chi_\rho(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \rho^k \cos kx = \frac{1 - \rho^2}{2\left(1 - 2\rho cosx + \rho^2\right)}. \tag{2}$$

Through A_{∞}^{h} (A_{1}^{h}) $(-\infty < h < \infty)$ denote the set off all functions that allow an analytic continuation to a function $f(z) = f(x+iy)$, analytic in the strip $|y| < h$, and such, that with all $|y| < h$ $\|\operatorname{Re} f(g+iy)\|_{\infty} \le 1$ ($\|\operatorname{Re} f(g+iy)\|_{1} \le 1$).

Classes of functions A_{∞}^{h} (A_{1}^{h}) (see, for example, [1, p. 186]) coincide with sets of functions that are convolutions

$$f(x) = \frac{1}{\pi}\int_{0}^{2\pi}\Psi_{h}(x-t)\varphi(t)\,dt, \qquad (3)$$

where the kernel $\Psi_{h}(x)$ has the following form

$$\Psi_{h}(x) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\cos kx}{ch\,kh}\ . \qquad (4)$$

In this case in correlations (1) та (3) $\|\varphi\|_{\infty} \le 1$ ($\|\varphi\|_{1} \le 1$) and $\int_{0}^{2\pi}\varphi(t)\,dt = 0$, i.e. function $\varphi(\bullet)$ belongs to single ballet of space L_{∞} (L), which we will further denote through U_{∞}^{0} (U_{1}^{0}).

The classes, that are considered, can be considered as special cases introduced (see, for example, [2, 3]) functional classes $C_{\beta,\infty}^{\psi}$ ($L_{\beta,1}^{\psi}$).

Through $\Sigma_{n}(\varphi; t_{n-1,i}; x)$, ; $i = 1, 2$, denote sum

$$\sum\nolimits_{n}(\varphi; t_{n-1,i}; x) = \Big(\big(\varphi * \chi_{\rho}\big)(x) - t_{n-1,1}(x)\Big) + \Big(\big(\varphi * \Psi_{h}\big)(x) - t_{n-1,2}(x)\Big),$$

where $*$ is the symbol of convolution of function χ_{ρ} and Ψ_{h} view (2) and (4) respectively with function $\varphi(g)$.

In this article, similarly to the reasoning presented in the works [4-6], the values $\hat{E}_{n,2}\left(U_{1}^{0}\right)_{L} = \sup\limits_{\varphi\in U_{1}^{0}}\inf\limits_{t_{n-1,i}}\|\Sigma_{n}(\varphi; t_{n-1,i}; x)\|_{L}$ — the best joint one sided approximation is investigated.

The purpose of the work. To get in the problem the exact constants one side approximation of the sum of functions given as convolutions (1), (3) in the metric of space *L*, namely, to find the exact values of the quantity

$$\hat{E}_{n,2}\left(U_{1}^{0}\right)_{L} = \sup_{\varphi\in U_{1}^{0}}\inf_{t_{n-1,i}}\|\Sigma_{n}(\varphi; t_{n-1,i}; x\|_{L}\ , \qquad (5)$$

where

$$t_{n-1,1}(x) \geq (\varphi * \chi_\rho)(x), \; t_{n-1,2}(x) \geq (\varphi * \Psi_h)(x), \; 0 \leq x \leq 2\pi, \qquad (6)$$

which we call the values of the best joint one sided approximation of classes Γ_1^ρ and A_1^h.

In addition to appearance limitations (6) in (5) sometime consider opposite limitations

$$t_{n-1,1}(x) \leq (\varphi * \chi_\rho)(x), \; t_{n-1,2}(x) \leq (\varphi * \Psi_h)(x), \; 0 \leq x \leq 2\pi, \qquad (6')$$

in a similar formulation of the problem.

Revelance of the topic. Exact values of the upper borders of the best approximations of classes A_∞^h i Γ_∞^ρ by trigonometric polynomials in uniform metric are obtained in the works of Ahiezer N. I. [7] and Crane M. G. [8]. Using this results, Nikolsky S. M. [9], obtained the exact upper limits for the best the approximations of classes A_1^h i Γ_1^ρ by trigonometric polynomials in integral metric by methods of dual ratios.

The exact values of the best joint approximations in uniform and integral metrics of the sum of functions from classes of harmonic functions and Poisson integrals where found by authors in [10]. Based on this results, we will find the value of the quantity (5) for any $n \in \mathbb{N}$.

Auxiliary statements. The Poisson kernel (see (2)) reaches the maximum value at points $x = 2m\pi\,(m = 0, \pm 1, \pm 2, \ldots)$, and the minimum — at the points $x = (2m+1)\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$. Denote by

$$g_n(f;t) = \frac{1}{n}\sum_{k=1}^{n} f\left(t + \frac{2k\pi}{n}\right). \qquad (7)$$

At work [10, p. 78] it is shown, that

$$\max_t g_n(\chi_\rho;t) = g_n(\chi_\rho;0) = \frac{1}{2}\frac{1+\rho^n}{1-\rho^n}, \qquad (8)$$

$$\min_t g_n(\chi_\rho;t) = g_n\left(\chi_\rho;\frac{\pi}{n}\right) = \frac{1}{2}\frac{1-\rho^n}{1+\rho^n}. \qquad (9)$$

Similarly, for the kernel $\Psi_h(x)$ of (4), for which the development into infinite product is known

$$\Psi_h(t) = \alpha\prod_{m=1}^{\infty}\left(\frac{2ch(2m-1)h}{2ch^2\left(m-\frac{1}{2}\right)h - (1+\cos t)} - 1\right) \; (\alpha > 0), \qquad (10)$$

we will have such extreme values

$$\max_t g_n(\Psi_h;t) = g_n(\Psi_h;0) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{ch\, nkh}, \tag{11}$$

$$\min_t g_n(\Psi_h;t) = g_n\left(\Psi_h;\frac{\pi}{n}\right) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{ch\, nkh}. \tag{12}$$

Let $x_k = \dfrac{2k\pi}{n}$ $(k = 0,1,\dots,n-1)$, $y_k = \dfrac{(2k+1)\pi}{n}$ $(k = 0,1,\dots,n-1)$. From the ratios (7)-(9) for the Poisson kernel $\chi_\rho(\bullet)$, and accordingly (7), (11), (12) for the kernel $\Psi_h(\bullet)$ we get, that $\sum_{k=0}^{n-1} \chi'_\rho(x_k) = 0$, $\sum_{k\,0}^{n-1} \Psi'_h(x_k) = 0$, $\sum_{k=0}^{n-1} \chi'_\rho(y_k)(y_k) = 0$, $\sum_{k=0}^{n-1} \Psi'_h(y_k) = 0$, and why and

$$\sum_{k=0}^{n-1}\left(\chi'_\rho(x_k) + \Psi'_h(x_k)\right) = 0,\ \sum_{k=0}^{n-1}\left(\chi'_\rho(y_k) + \Psi'_h(y_k)\right) = 0.$$

From work [7, p. 120] we obtained, that there are only trigonometric polynomials $t^+_{n-1,1}(\chi_\rho;x)$, $t^-_{n-1,1}(\chi_\rho;x)$, $t^+_{n-1,2}(\Psi_h;x)$, $t^-_{n-1,2}(\Psi_h;x)$ order not higher than $n-1$, for which equalities hold

$$t^+_{n-1,1}(\chi_\rho;x_k) = \chi_\rho(x_k),\quad t^{+\prime}{}_{n-1,1}(\chi_\rho;x_k) = \chi'_\rho(x_k),$$

$$t^-_{n-1,1}(\chi_\rho;y_k) = \chi_\rho(y_k),\quad t^{-\prime}{}_{n-1,1}(\chi_\rho;y_k) = \chi'_\rho(y_k)\quad (k = 0,1,\dots,n-1),$$

$$t^+_{n-1,2}(\Psi_h;x_k) = \Psi_h(x_k),\quad t^{+\prime}{}_{n-1,2}(\Psi_h;x_k) = \Psi'_h(x_k),$$

$$t^-_{n-1,2}(\Psi_h;y_k) = \Psi_h(y_k),\quad t^{-\prime}{}_{n-1,2}(\Psi_h;y_k) = \Psi'_h(y_k)\quad (k = 0,1,\dots,n-1).$$

Let

$$t^+_{n-1}(\chi_\rho;\Psi_h;x) = t^+_{n-1,1}(\chi_\rho;x) + t^+_{n-1,2}(\Psi_h;x),$$

$$t^-_{n-1}(\chi_\rho;\Psi_h;x) = t^-_{n-1,1}(\chi_\rho;x) + t^-_{n-1,2}(\Psi_h;x).$$

Lemma 1. Functions $\chi_\rho(x) + \Psi_h(x) - t^+_{n-1}(\chi_\rho;\Psi_h;x)$ and $\chi_\rho(x) + \Psi_h(x) - t^-_{n-1}(\chi_\rho;\Psi_h;x)$ acquire the values of one sing throughout the interval $[0;2\pi]$.

Proving the validity of the lemma, we will repeat reasoning of Tikhomirov V. M. [1, p. 189]. Put $\Delta_1(x) = \chi_\rho(x) - t^+_{n-1,1}(\chi_\rho;x)$, $\Delta_2(x) = \Psi_h(x) - t^+_{n-1,2}(\Psi_h;x)$.

Suppose a function $\Delta_1(x)$ change sign on $[0;2\pi]$. Obviously, the difference $\Delta_1(x)$ is an even function, and therefore, due to parity, it changes the sign inside the intervala $(0;\pi)$. We get, that $\delta(t)=\Delta_1(\arccos t)$ has at least $(n+1)$ zero on the interval $[-1\,;1]$, if each of zero lying inside the interval $(-1;1)$, take into account so many times what its multiplicity is, and the zeros at the end of the interval (if there are any) are counted only once. Applicable to the function $\delta(t)$ n times Rolle's theorem and we obtain that function $\delta^{(n)}(t)$ has at least one zero in interval $[-1;1]$. And this is impossible, because $\delta^{(n)}(t)=\dfrac{(2\rho)^n\left(1-\rho^2\right)n!}{2\left(1-2\rho t+\rho^2\right)^{n+1}}$. And, therefore, $\delta^{(n)}(t)>0$, $t\in[-1;1]$. They came to contradiction. So, the function $\Delta_1(x)$ does not change the sing on the period. For a function $\chi_\rho(x)-t^{-}_{n-1,1}$ constancy of sign established similarly.

Consider now the difference $\Delta_2(x)$. Since the function $\Psi_h(x)$ is even, the polynomial — $t^{+}_{n-1,2}(\Psi_h;x)$ is even too. From equality (10) it follows, that all derivatives of function

$$\Psi_h(\arccos t)=\alpha\prod_{m=1}^{\infty}\left(\frac{2ch(2m-1)h}{2ch^2\left(m-\frac{1}{2}\right)h-(1+\cos t)}-1\right),\ \alpha>0,$$

are positive on interval $[-1;1]$. This means that the function $\left(\Psi_h(\arccos t)-t_{n-1,2}(\arccos t)\right)^{(n)}>0$, $t\in[-1;1]$. So, the function $\Delta_2(x)$ is one sing on period also. For a function $\Psi_h(x)-t^{-}_{n-1,2}(\Psi_h;x)$ constancy of sign established similarly.

Since the functions $\Delta_1(x)$ and $\Delta_2(x)$ on the period and at zero have the same sing, the lemma hold for the function $\chi_\rho(x)+\Psi_h(x)-t^{+}_{n-1}(\chi_\rho;\Psi_h;x)$. Similarly, lemma takes place for a function $\chi_\rho(x)+\Psi_h(x)-t^{-}_{n-1}(\chi_\rho;\Psi_h;x)$.

Main result.

Theorem. Let $0 < \rho < 1$, $-\infty < h < \infty$, n = 1, 2, …, then fair equality

$$\hat{E}_{n,2}\left(U_1^0\right)_L = 2\rho^n\left(1- \rho^n\right)^{-1} + 2\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}. \tag{13}$$

Proof. To find the exact values of the lower limits of the best one side approximations of the sum of majorant functions from classes Γ_1^{ϱ} and A_1^h, we establish some equalities.

Lemma 2. The following correlations are valid:

$$\hat{E}_{n,2}\left(\chi_\rho + \Psi_h\right)_L =$$

$$= \inf_{\substack{t_{n-1,1}(\bullet)\ge(\varphi*\chi_\rho)(\bullet)\\ t_{n-1,2}(\bullet)\ge(\varphi*\Psi_h)(\bullet)}} \left\|(\chi_\rho + \Psi_h)(\bullet) - (t_{n-1,1} + t_{n-1,2})(\bullet)\right\|_1 = \tag{14}$$

$$= 2\pi\rho^n\left(1-\rho^n\right)^{-1} + 2\pi\sum_{k=1}^{\infty}\frac{1}{ch\,nkh};$$

$$\breve{E}_{n,2}\left(\chi_\rho + \Psi_h\right)_L =$$

$$= \inf_{\substack{t_{n-1,1}(\bullet)\le\left(\varphi*\chi_\rho\right)(\bullet)\\ t_{n-1,2}(\bullet)\le\left(\varphi*\Psi_h\right)(\bullet)}} \left\|(\chi_\rho + \Psi_h)(\bullet) - (t_{n-1,1} + t_{n-1,2})(\bullet)\right\|_1 = \tag{14′}$$

$$= 2\pi\rho^n\left(1+\rho^n\right)^{-1} + 2\pi\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{ch\,nkh}.$$

Proof of lemma. Let $t_{n-1,1}(x)$, $t_{n-1,2}(x)$) — arbitrary trigonometric polynomials of order not higher than n – 1, different from $t_{n-1,1}^{+}\left(\chi_\rho; x\right)$ (respectively $t_{n-1,2}^{+}\left(\Psi_h; x\right)$), and such that for all $x\in[0;2\pi]$ $t_{n-1,1}(x)-\chi_\rho(x)\ge 0$ and $t_{n-1,2}(x)-\Psi_h(x)\ge 0$. Due to the only one of trigonometric polynomial $t_{n-1}^{+}\left(\chi_\rho;\Psi_h;x\right)$ order not higher than n – 1, we will have that at least one point $x_k\in[0;2\pi]$ inequality

$$\left(t_{n-1,1}(x_k)+t_{n-1,2}(x_k)\right)-\left(\chi_\rho(x_k)+\Psi_h(x_k)\right) > 0$$

is satisfied. And therefore, taking into account (7), we get

$$g_n(t_{n-1,1} + t_{n-1,2};0) - g_n(\chi_\rho + \Psi_h;0) =$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(t_{n-1,1} + t_{n-1,2}\right)(0 + \frac{2k\pi}{n}) - \frac{1}{n}\sum_{k=1}^{n}(\chi_\rho + \Psi_h)(0 + \frac{2k\pi}{n}).$$

Taking into consideration that for any trigonometric polynomial $t_{n-1}(\bullet)$ of order not higher than n – 1 function $g_n(t_{n-1};x) \equiv \text{const}$, the correlation

$$g_n(t_{n-1,1}+t_{n-1,2};x) \equiv g_n\left(t_{n-1,1}+t_{n-1,2};0\right) \tag{15}$$

takes place.

Using the formula (7), we are also convinced that the equalities are true:

$$g_n(\chi_\rho;x) = \frac{1}{2}+\sum_{k=1}^{\infty}\rho^{nk}\cos nkx = \chi_{\rho^n}\left(nx\right), \tag{16}$$

$$g_n(\Psi_h;x) = \frac{1}{2}+\sum_{k=1}^{\infty}\frac{\cos nkx}{ch\,nkh} = \Psi_{nh}\left(nx\right). \tag{17}$$

In force (15), from correlations (16), (17) we get

$$\begin{gathered}\int_0^{2\pi}\left[\left(t_{n-1,1}+t_{n-1,2}\right)(x)-\left(\chi_\rho+\Psi_h\right)(x)\right]dx = \\ = \int_0^{2\pi}\left[g_n\left(t_{n-1,1}+t_{n-1,2};x\right)-g_n\left(\chi_\rho+\Psi_h;x\right)\right]dx > \\ > \int_0^{2\pi}\left[g_n\left(\chi_\rho+\Psi_h;0\right)-g_n\left(\chi_\rho+\Psi_h;x\right)\right]dx = \\ = 2\pi\rho^n\left(1-\rho^n\right)^{-1}+2\pi\rho^n\left(1+\rho^n\right)^{-1}.\end{gathered} \tag{18}$$

Since

$$g_n(t_{n-1,1}+t_{n-1,2};x) \equiv g_n\left(t_{n-1,1}+t_{n-1,2};0\right) = g_n(\chi_\rho+\Psi_h;0),$$

then

$$\begin{gathered}\int_0^{2\pi}\left[\left(t^+_{n-1,1}+t^+_{n-1,2}\right)(x)-\left(\chi_\rho+\Psi_h\right)(x)\right]dx = \\ = \int_0^{2\pi}\left[g_n\left(t_{n-1,1}+t_{n-1,2};x\right)-\left(\chi_\rho+\Psi_h\right)(x)\right]dx = \\ = \int_0^{2\pi}\left[g_n\left(t_{n-1,1}+t_{n-1,2};x\right)-g_n\left(\chi_\rho+\Psi_h;x\right)\right]dx = \\ = \int_0^{2\pi}\left[g_n\left(\chi_\rho+\Psi_h;0\right)-g_n\left(\chi_\rho+\Psi_h;x\right)\right]dx = \\ = 2\pi\rho^n\left(1-\rho^n\right)^{-1}+2\pi\rho^n\left(1+\rho^n\right)^{-1}.\end{gathered} \tag{19}$$

Taking into account lemma 1 and comparing the correlations (18), (19), we are convinced of the fairness of equality (14). Equality (14′) is installed similarly.

We continue to prove the theorem. When proving equality (13) for the upper border of the best approximation on the function class U_1^0 we consider any function $\varphi(x) \in U_1^0$. By $f_1(x)$ denote the convolution of that function with Poisson kernel, and by $f_2(x)$ — the convolution with kernel Ψ_h. Denote $\varphi_+(x) = \max\left[\varphi(x), 0\right]$ and accordingly $\varphi_-(x) = \max\left[-\varphi(x), 0\right]$. Let's choose a trigonometric polynomial of this form

$$t_{n-1}(f;x) = t_{n-1}^*(f_1 + f_2 \ ;x) = \frac{1}{\pi}\int_0^{2\pi} t_{n-1}^*\left(\chi_\rho + \Psi_h; x-t\right)\varphi_+(t)\,dt -$$

$$-\frac{1}{\pi}\int_0^{2\pi} t_{n-1}^*\left(\chi_\rho + \Psi_h; x-t\right)\varphi_-(t)\,dt,\ f_1(\bullet) \in \Gamma_1^\rho,\ f_2(\bullet) \in A_1^h.$$

Obviously,

$$t_{n-1}^*(f_1 + f_2 \ ;x) - (f_1 + f_2)(x) =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left[t_{n-1}^*\left(\chi_\rho + \Psi_h \ ;x-t\right) - \left(\chi_\rho + \Psi_h\right)(x-t)\right]\varphi_+(t)\,dt +$$

$$+\frac{1}{\pi}\int_0^{2\pi}\left[\left(\chi_\rho + \Psi_h\right)(x-t) - t_{n-1}^-\left(\chi_\rho + \Psi_h; x-t\right)\right]\varphi_-(t)\,dt > 0.$$

And for each sum of functions $(f_1 + f_2)(\bullet) \in \left(\Gamma_1^\rho + A_1^h\right)$ we will have

$$\hat{E}_{n,2}(f_1 + f_2)_L \le \left\|t^*{}_{n-1}(f_1 + f_2) - (f_1 + f_2)\right\|_L \le$$

$$\le \frac{1}{\pi}\left\|\varphi_+\right\|_L \left\|t_{n-1}^+(\chi_\rho + \Psi_h) - (\chi_\rho + \Psi_h)\right\|_L +$$

$$+\left\|\varphi_-\right\|_L \left\|t_{n-1}^-(\chi_\rho + \Psi_h) - (\chi_\rho + \Psi_h)\right\|_L = \frac{1}{\pi}\left\|\varphi_+\right\|_L (\left(2\pi\rho^n\right)\left(1-\rho^n\right)^{-1} +$$

$$+2\pi\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}) + \frac{1}{\pi}\left\|\varphi_-\right\|_L (\left(2\pi\rho^n\right)\left(1+\rho^n\right)^{-1} + 2\pi\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{ch\,nkh}) \le$$

$$\le \left(\frac{2\rho^n}{1-\rho^n} + 2\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}\right)\left(\left\|\varphi_+\right\|_L + \left\|\varphi_-\right\|_L\right) \le 2\rho^n\left(1-\rho^n\right)^{-1} + 2\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}$$

So, we got an estimate from above of the value $\hat{E}_{n,2}\left(U_1^0\right)_L$ in task (5), namely

$$\hat{E}_{n,2}\left(U_1^0\right)_L \le 2\rho^n\left(1-\rho^n\right)^{-1}+2\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}. \qquad (20)$$

For any $\theta>0$ let's choose the following functions $f_{1_\theta}(x)$, $f_{2_\theta}(x)$:

$$f_{1_\theta}(x)=\frac{1}{\pi}\int_0^{2\pi}\chi_\rho(x-t)\delta_\theta(t)\,dt\,,\quad f_{2_\theta}(x)=\frac{1}{\pi}\int_0^{2\pi}\Psi_h(x-t)\delta_\theta(t)\,dt\,,$$

where $\delta_\theta(t)$ — 2π-periodic function, that coincide with the value $\frac{1}{\theta}$, if $|t|\le\frac{\theta}{2}$, and is equal to 0, if $\frac{\theta}{2}<|t|\le\pi$. With values θ: $0<\theta<\pi$, $\delta_\theta(t)\in U_1^0$, therefore functions $f_{1_\theta}(x)$ and $f_{2_\theta}(x)$ will belong to the classes accordingly: $f_{1_\theta}(x)\in\Gamma_1^\rho$, $f_{2_\theta}(x)\in A_1^h$. And for such a sum of functions we will have

$$\hat{E}_{n,2}\left(f_{1_\theta}(x)+f_{2_\theta}(x)\right)_L=\frac{1}{\pi}\hat{E}_{n,2}\left(\frac{1}{\theta}\int_{x-\frac{\theta}{2}}^{x+\frac{\theta}{2}}\left(\chi_\rho+\Psi_h\right)(t)\,dt\right)_L=$$

$$=\frac{1}{\pi}\hat{E}_{n,2}\left(\frac{1}{\theta}\int_{x-\frac{\theta}{2}}^{x+\frac{\theta}{2}}\left(\chi_\rho+\Psi_h\right)(t)\,dt-\left(\chi_\rho+\Psi_h\right)(x)+\left(\chi_\rho+\Psi_h\right)(x)\right)_L=$$

$$=\frac{1}{\pi}\hat{E}_{n,2}\left(\chi_\rho+\Psi_h\right)(x)_L+o(\theta)=2\rho^n\left(1-\rho^n\right)^{-1}+2\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}+o(\theta).\,(21)$$

If we will go to limit in (21) under the condition $\theta\to 0$, then we will obtain the inequality

$$\hat{E}_{n,2}\left(U_1^0\right)_L \ge \lim_{\theta\to 0}\hat{E}_{n,2}\left(f_{1_\theta}(x)+f_{2_\theta}(x)\right)_L=$$
$$=2\rho^n\left(1-\rho^n\right)^{-1}+2\sum_{k=1}^{\infty}\frac{1}{ch\,nkh}. \qquad (22)$$

If we compare the ratios (20) and (22), then we will be convinced of the correctness of the theorem.

The given work can be extended to the case of an arbitrary the number of terms in the linear combination $\sum_{n,m}(\varphi;t_{n-1,i};x)$, $i=\overline{1;m}$.

Conclusion. The exact value of the value of the best joint one sided approximation above the sum of functions from the classes of harmonic functions in the circle and Poisson integrals by trigonometric polynomials of order no higher than $n-1$ in integral metric for each $n \in N$ is found.

Remark. Applying similar consideration, we can solve the problem of the best joint one sided approximation the sum of functions from the classes of harmonic functions in the circle and Poisson integrals by trigonometric polynomials of order no higher than $n-1$ in integral metric from below, that is the problem of finding the exact value of the quantity

$$\breve{E}_{n,2}\left(U_1^0\right)_L = \sup_{\varphi \in U_1^0} \inf_{t_{n-1,i}} \left\| \Sigma_n(\varphi; t_{n-1,i}; x) \right\|_L ,$$

where

$$t_{n-1,1}(x) \le \left(\varphi * \chi_\rho\right)(x),\ t_{n-1,2}(x) \le \left(\varphi * \Psi_h\right)(x),\ 0 \le x \le 2\pi .$$

If the conditions of the theorem are fulfilled, the equality

$$\breve{U}_n\left(U_1^0\right)_L = 2\rho^n\left(1+\ \rho^n\right)^{-1} + 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{ch\, nkh}.$$

is valid for each $n \in N$.

References:

1. Tikhomirov V. M. Some questions of the theory of approximations. Moscow: Publishing house of the MSU. 1976. 304 p.
2. Stepanets A. I. Classification and approximation of periodic functions. Kiev: Naukova dumka. 1987. 268 p.
3. Stepanets A. I. Methods of Approximation Theory. Kiev: Institute of Mathematics of the National Academy of Sciences of Ukraine, 2002. Part I. 427 p.
4. Ganelius T. On one sided approximation by trigonometric polynomials. *Mathematica Scandinavica.* 1956. Vol 4. № 2. P. 247-258.
5. Sorich V. A., Sorich N. M. On the best one side sided approximation of classes W_β^r in metric *L*. *Fourier series: The Theory and Application*: Collection the scientific works. Kiev, 1998. Vol. 20. P. 316-322.
6. Doronin V. G., Ligun A. A. About the best one sided approximation of classes $W_\alpha^r V \ \ (r > -1)$ by trigonometric polinomials in metric L_1. *Matematical notes.* 1977. Vol. 22. № 3. P. 357-370.
7. Ahiezer N. I. Lectures on the theory of approximation. Moscow: Nauka, 1965. P. 407.
8. Crane M. G. To the theory of the best approximation of periodic functions. *Reports of the Academy of Sciences USSR.* 1938. Vo. 18. № 4-5. P. 245-249.
9. Nikolsky S. M. Approximation of functions be trigonometric polynomials on the average. *Information of the Academy of Sciences USSR. Math. series.* 1946. Vo. 10. P. 207-256.
10. Sorich V. A., Sorich N. M. Exact constants of the best approximations of the sum functions of analytic functions from different classes. *Herald of Kamianets-Podilskyi Ivan Ohiienko National University. Phisical and math. Series.* 2019. Vol. 12. P. 76-81.

ТОЧНІ КОНСТАНТИ НАЙКРАЩИХ ОДНОСТОРОННІХ НАБЛИЖЕНЬ СУМИ АНАЛІТИЧНИХ ФУНКЦІЙ ІЗ РІЗНИХ КЛАСІВ

Задача отримання точних значень найкращих наближень тригонометричними многочленами неперервних або сумовних функцій бере свій початок у роботах П. Л. Чебишева, який ще у 50-х роках XIX століття поставив задачу про знаходження многочлена, який найменше відхиляється від заданої неперервної функції. Згодом цей напрям в теорії наближення набув подальшого розвитку завдяки роботам К. Вейєрштрасса, Д. Джексона, С. Н. Бернштейна, Валле-Пуссена та ін. На даний час спостерігається підвищена увага до задач одностороннього наближення окремих функцій та їх класів в метриці простору L. Задачі такого змісту виникають в теорії чисел, теорії кодування та інших областях математики. Перші результати такого напрямку були отримані в 1880-х роках А. А. Марковим та Т. Й. Стілтьєсом. В подальшому ці дослідження були продовжені в роботах Й. Карамати (1930), Г. Фройда і Т. Ганеліуса (середина XX століття).

Загальні питання, пов'язані із задачею найкращого наближення класів функцій тригонометричними многочленами: існування многочлена найкращого наближення, його характеристичних властивостей, односторонні наближення детально викладені у багатьох працях, зокрема, наприклад, в книзі М. П. Корнєйчука [1], працях Т. Ганеліуса [4], В. Г. Дороніна, А. А. Лігуна [5].

В даній роботі знайдено точні константи найкращого одностороннього наближення суми мажорантних функцій класів, що допускають аналітичне продовження в смугу фіксованої ширини, та функцій, гармонійних в крузі радіуса 1.

Ключові слова: *гармонійні в крузі функції, аналітичні в смузі функції, найкраще сумісне одностороннє наближення.*

Отримано: 11.10.2022