such composite kernels that their trigonometric polynomial of the best approximation of order $n - 1$ in the integral metric interpolates the kernel at $2n + 2$ uniformly distributed points.

**Key words:** *best approximation, integral metric, linear combination of kernels of different parity.*

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# A MODEL OF ALCOHOL CONSUMPTION WITH SEMI-MARKOV VARIABLE COEFFICIENTS

This paper focuses on the in-depth study of a stochastic approximation methodology involving semi-Markov switches in an averaging scheme with a minor parameter. We present a model where perceived objects are impacted by noise variables dependent on the semi-Markov process. The emphasis is on analyzing the convergence and stability of the stochastic approximation process within this context. Theoretical results are presented alongside numeric simulations, demonstrating the practical applications in managing complex stochastic systems. This work opens promising paths for the use of stochastic approximation approaches in the wider field of semi-Markov processes. Recognizing the mutable nature of alcohol consumption and its dependency on a variety of factors, we propose encoding such dynamism within a semi-Markov parameter structure. This model handles the patterns of individual alcohol consumption as a semi-Markov process; the transition probabilities between different states of alcohol consumption are directed by sociodemographic variables that change over time. Our approach, thus, bridges the gap between the realities of ever-changing alcohol consumption trends and static traditional Markov-chain models. By integrating real-world variables into our innovative model, we offer a cutting-edge analytical tool that lays down new paths for understanding and addressing the challenges with alcohol consumption patterns. Further, our insights have the potential to significantly impact the formulation of effective strategies and public health interventions aimed at alcohol-related harm reduction.

**Key words**: *semi-Markov processes, diffusion processes, Wiener process, stochastic models, stochastic processes.*

 

**1. Introduction.** Due to the wide use of stochastic diffusion problems, conditions for stability and control of such systems seem to be important. In [7] sufficient conditions for stability of stochastic systems via Lyapunov function properties are given and estimates of large deviations of linear diffusion systems are obtained. Problems of optimal control of diffusion processes described by stochastic differential equations with acceptable control are described in [21]. On the other hand, asymptotic behavior is considered in [25] and [26]. For conditions of weak convergence of random processes in [12, 17, 13] the method of small parameter and a singular perturbation problem solution was used for the construction of the generator limiting process. This method is applied to schemes of averaging of diffusion approximation and to schemes of averaging of asymptotically small diffusion. In [17] the cases of random evolution of Markov and semi-Markov switching were examined. Construction of semi-Markov processes and investigation of asymptotic properties of random processes with semi-Markov switching are considered in [2-5]. For these processes, weak convergence to the solution of appropriate partial differential equations and an averaging scheme of diffusion processes in diffusion 0approximation is presented in [4, 6]. In [20] asymptotic properties of semi-Markov processes with linearly perturbed operator maintainer a Markov process was analyzed via the semi group property. These results were then developed in [15]. A classification of solutions of a singular perturbation problem for random processes with the use of semi-Markov switching is described in [17] and in [16] with the use of the compensating operator (see [28]). Using the compensating operator [11] one obtains sufficient conditions for convergence of a random evolution with semi-Markov switching to the diffusion process in the averaging scheme (see also [14]). The results of these studies have found various applications [9, 10, 18, 19]. In [22] convergence of stochastic procedures is established using properties of Lyapunov type functions. Stochastic Approximation Procedure (SAP) by a regression function with semi-Markov switching was considered in [1, 8].

**2. Problem.** In this paper, we consider a dynamical system with semi-Markov switching using a small parameter $(x(t), t>0)$ , is a semi-Markov process in the standard phase space $(X,\xi)$, generated by the renewal Markov process $x_n$, $\tau_n$, $n \geq 0$, defined by the semi-Markov kernel:

$$Q(t,x,B) = P(x,B)G_x(t),$$

where the stochastic kernel

$$P(x,B) := P\{x_{n+1} \in B \mid x_n = x\}$$

defines an embedded Markov chain $x_n = x(\tau_n)$ , at the renewal moments,

$$\tau_n = \sum_{k=1}^{n} \theta_k, n \geq 0, \tau_0 = 0,$$

with intervals $\theta_{k+1} = \tau_{k+1} - \tau_k$ between the renewal moments. The $\theta_n$ are defined by the distribution functions:

$$G_x(t) = P\{\theta_{n+1} \leq t \mid x_n = x\} =: P\{\theta_{n+1}\{x\} \leq t\}.$$

Define

$$g(x) = \int_0^{\infty} \overline{G_x}(ds), \overline{G_x(s)} = 1 - G_x(s)$$

the semi-Markov process is defined by:

$$x(t) = x_{\nu(t)},$$

where the counting process $\nu(t)$ is defined by

$$\nu(t) := \max\{n : \tau_n \leq t\}, t \geq 0.$$

We shall assume that the semi-Markov process $x(t), r \geq 0$ is regular (the probability of reaching any state is positive) and uniformly ergodic with stationary distribution $\pi(B), B \in \xi$

$$\pi(dx) = \rho(dx) g(x) / m, m = \int_X g(x) \rho(dx).$$

Here $\rho(B), B \in \xi$, is a stationary distribution of the embedded Markov chain $(x_n)$. Note that the process $x(t)$ has a generator $\boldsymbol{Q}$:

$$\boldsymbol{Q}\varphi(x) = 1/g(x) \int_X P(x, dy)[\varphi(y) - \varphi(x)],$$

which acts in the Banach space $\boldsymbol{B}$ of all bounded real-valued measurable functions on $X$, with the sup-norm $\|\varphi\| = \sup_{x \in X} |\varphi(x)|$ for $\varphi \in \mathbf{B}$. We have

$$\boldsymbol{B} = \boldsymbol{N_Q} \oplus \boldsymbol{R_Q}, \tag{1}$$

where $N_{\boldsymbol{Q}} := \{\varphi : \boldsymbol{Q}\varphi = 0\}$ and $R_{\boldsymbol{Q}} := \{\psi : \boldsymbol{Q}\varphi = \psi\}$. Given **Q**, we can define the potential operator or simply the potential of **Q** by

$$R_0 = \Pi - (\boldsymbol{Q} + \Pi)^{-1},$$

where

$$\Pi\varphi(x) := \int_X \pi(dx) \varphi(x).$$

SAP [1] for a diffusion process $u^{\varepsilon}(t) \in \boldsymbol{R}^d$ in an averaging scheme with a small parameter $\varepsilon \geq 0$ is defined by a stochastic differential equation:

$$du^{\varepsilon}(t)=a(t)\Big[C\big(u^{\varepsilon}(t);x(t/\varepsilon)\big)dt+\sigma\big(u^{\varepsilon}(t);x(t/\varepsilon)\big)dw(t)\Big], \quad (2)$$

where: $u^{\varepsilon}(t), t\ge 0$, is a random evolution in a diffusion process, $x(t), t\ge 0$, is a semi-Markov process, $w(t)$ – Wiener process, $a(t)$ satisfies the following conditions [1] $\int_{t_0}^{\infty} a(t)dt=\infty, \int_{t_0}^{\infty} a^2(t)dt\le\infty$.

The semi group $\boldsymbol{T}_x(t), t\ge 0, x\in X$, associated to the system:

$$du_x(t)=a(t)\big[C(u_x(t);x)+\sigma(u_x(t);x)dw(t)\big], u_x(0)=u, \quad (3)$$

is defined by:

$$\boldsymbol{T}_x(t)\varphi(u)=\varphi(u_x(t,u)), \quad (4)$$

where:

$$u_x(t,u):=u_x(t), u_x(0)=u.$$

Notice that $u_x(t+s,u)=u_x(s,u_x(t,u))$, which is the semi group property for the trajectories $u_x(t,u)$. The generating operator $\boldsymbol{A}_x(t)$ of the semi group $\boldsymbol{T}_x(t)$ is defined by:

$$\boldsymbol{A}_x(t)\varphi(u)=a(t)C(u,x)\varphi'(u)+a^2(t)1/2\sigma^2(u,x)\varphi''(u),$$

where $\varphi(u)\in C^2(\boldsymbol{R}^d)$ and $a(t)\in\boldsymbol{C}^1(\boldsymbol{R})$. Note that a solution of (3) exists when the following conditions are satisfied:

$$\|C(u_1,x)-C(u_2,x)\|+\|\sigma(u_1,x)-\sigma(u_2,x)\|<l(\|u_1-u_2\|),$$

for anyone $x\in X$,

$$\|C(u,x)\|+\|\sigma(u,x)\|<l(1+\|u\|),$$

for anyone $x\in X$, for some $l>0$.

**3. Application.** This result can be used in controlling the process in an averaging scheme with semi-Markov switching. A mathematical model for alcohol consumption is considered in an equilibrium point was obtained in the case of deterministic values of coefficients in the form:

$$\begin{cases} a'=\mu+\gamma-\gamma m+\beta\alpha^2-a\left[\beta+\mu+\gamma-d-d_A\right](1-a) \\ m'=\beta a-\beta a^2-m\left(\alpha+\mu+a\left(d-d_A\right)\right) \end{cases}, \quad (5)$$

where $\alpha$ =0.000110247 (rate at which a non risk consumer moves to the risk consumption sub population), $\beta$ =0.0284534 (transmission rate due to social pressure to increase alcohol consumption: family, friends, marketing, TV, etc.), $\mu$ =0.01 (birth rate), $dA$=0.008 (death rate), $d = 0.009$ (augmented death rate due to alcohol consumption), $\gamma$ =0.00144 (rate at which a risk consumer becomes a non-consumer), $a(t)$ is the rate of non-consumers, and $m(t)$ the rate of non-risk consumers. It is the same model as presented in [23], but without delay:

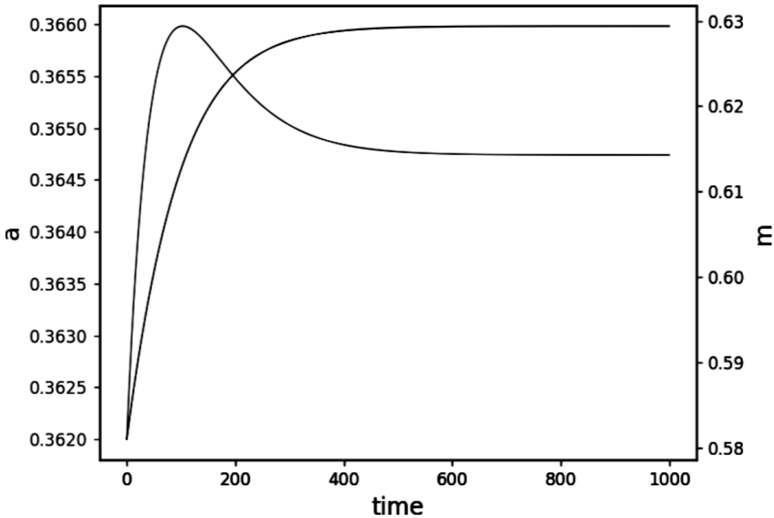


***Fig. 1**. Model*

The solution of system (11) converges to the equilibrium point: $a^* = 0.3647389407$, $m^* = 0.6293831151$. We propose below a modification of the above system considering a semi-Markov model with switching. More precisely, we consider two modifications studied in the following subsections.

**3.1. Model with stochastic drift.**

$$\begin{cases} a' = \mu + \gamma - \gamma m + \beta\alpha^2 - a[\beta + \mu + \gamma - d - d_A](1-a) + \sigma_1\left(a - a^*\right)\sigma_1 \\ m' = \beta a - \beta a^2 - m\left(\alpha + \mu + a\left(d - d_A\right)\right) + \sigma_2\left(m - m^*\right)\sigma_2'' \end{cases}, \quad (6)$$

where $\alpha$ =0.000110247, $\beta$ =0.0284534, $\mu$ =0.01, $dA$=0.008, $d$=0.009, $\gamma$ =0.00144, $a_0 = 0.362$, $m_0 = 0.581$, $a(t)$ is the non-consumers rate, and $m(t)$ the rate of non risk consumers. It is the same model as in [24], $I(a_t) = a_t$. Here $w_1', w_2''$ denote mutually independent standard Wiener processes.

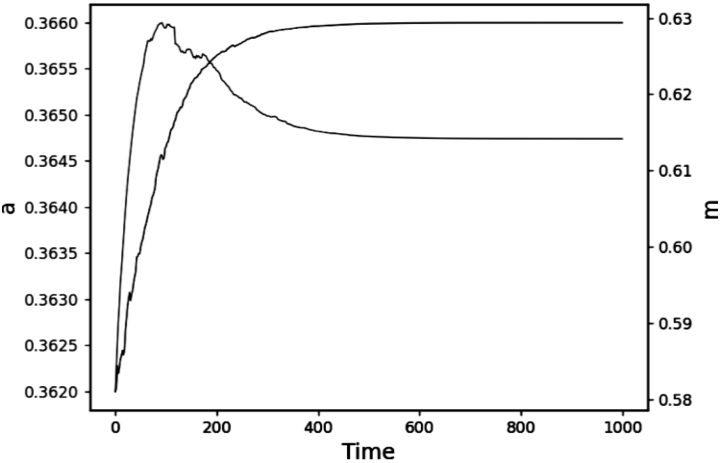


***Fig. 2***. *Model with stochastic drift*

The solution of (2) converges to the equilibrium point $a^* = 0.3647389407$, $m^* = 0.6293831151$.

**3.2. Stochastic approximation procedure applied to stochastic drift with semi-Markov switching.** The next model involves non constant coefficients. In this case, we use a stochastic approximation procedure for a diffusion process with semi-Markov switching:

$$\begin{cases} a' = c(t)\Big[\mu_X + \gamma - \gamma m + \beta\alpha^2 - a\left[\beta + \mu_X + \gamma - d - d_A\right]\times \\ \times(1-a) + \sigma_1\left(a - a^*\right)w_1'\Big], \\ m' = c(t)\Big[\beta a - \beta a^2 - m\big(\alpha + \mu_X + a\left(d - d_A\right)\big) + \sigma_2\left(m - m^*\right)w_2'\Big] \end{cases} \tag{7}$$

where $\alpha = 0.000110247$, $\beta = 0.0284534$, $\mu = 0.01$, $dA = 0.008$, $d = 0.009$, $\gamma = 0.00144$, $a_0 = 0.362$, $m_0 = 0.581$, $a(t)$ is the rate of non-consumers, and $m(t)$ the rate of non risk consumers. It is the same model as in [24], $I(a_t) = a_t$, $a^* = 0.3647389407$, $m^* = 0.6293831151$ (the solution of the simple model (2)). Here $w_1', w_2'$ are mutually independent standard Wiener processes and

$$\mu_X = \mu + X, \tag{8}$$

where $X$ is a semi-Markov process with states ±0.005, and with $G_x(t)$ uniformly distribution. $G_x(t)$ could have another distribution but should satisfy the Cramer condition (see (C6) of Theorem 3.1) [1]. Implementation of procedure (7) using the Python programming language in the PyCharm environment gives us the following results:

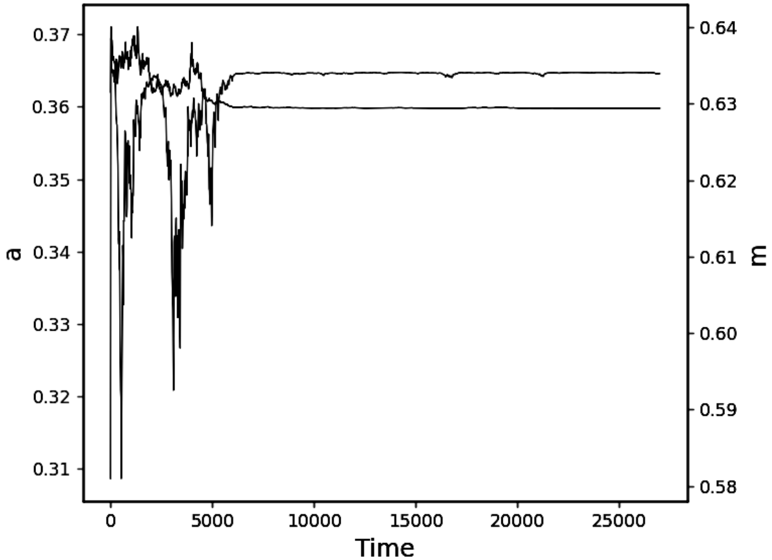


***Fig. 3.*** *Realization procedure 3.2 with stochastic drift with semi-Markov switching*

Note that additional disturbances are caused by the instability of the $\mu_X$ coefficient given in (8).

**3.3 Stochastic approximation procedure applied to stochastic drift with semi-Markov switching with two perturbed parameters.** Let's start form the fact that in addition to the $\mu$ we will disturb the $\gamma$ parameter, but the semi-Markov process disturbing this parameter will have values $\pm 0.0006$ and otherwise this process will be the same as the process disturbing $\mu$. We will then receive system:

$$\begin{cases} a' = c(t)[\mu_{X_1} + \gamma_{X_2} - \gamma_{X_2} m + \beta\alpha^2 - a\left[\beta + \mu_{X_1} + \gamma_{X_2} - d - d_A\right] \times \\ \times(1-a) + \sigma_1^{'}\left(a - a^*\right) w_1'] \\ m' = c(t)\left[\beta a - \beta a^2 - m\left(\alpha + \mu_{X_1} + a\left(d - d_A\right)\right) + \sigma_2\left(m - m^*\right) w_2'\right] \end{cases}, \quad (9)$$

where $\alpha = 0.000110247$, $\beta = 0.0284534$, $\mu = 0.01$, $dA = 0.008$, $d = 0.009$, $\gamma = 0.00144$, $a_0 = 0.362$, $m_0 = 0.581$, $a(t)$ is the rate of non-consumers, and $m(t)$ the rate of non risk consumers. It is the same model as in [24], $I(a_t)=a_t$, $a^*=0.3647389407$, $m^*=0.6293831151$ (the solution of the simple model (2)). Here $w_1', w_2'$ are mutually independent standard Wiener processes and

$$\begin{cases} \mu_{X_1} = \mu + X_1 \\ \gamma_{X_2} = \gamma + X_2 \end{cases}, \tag{10}$$

where $X_1$ is a semi-Markov process with states ±0.005, and with $G_x(t)$ uniformly distribution and $X_2$ is a semi-Markov process with states ±0.005, and with $H_x(t)$ normal distribution.

Implementation of procedure (9) using the Python programming language in the PyCharm environment gives us the following results:

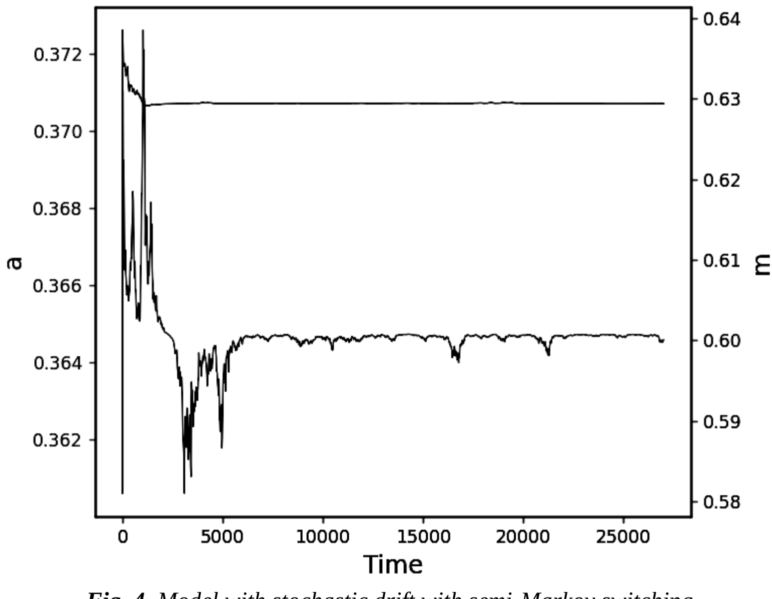


***Fig. 4.*** *Model with stochastic drift with semi-Markov switching with two perturbed parameters*

We can notice that the parameter $\gamma$ does not have a large impact on these processes, because its the disturbance did not cause such major changes. Of course, they aren't these are the same curves as for models with one perturbed parameter $\mu$, but these differences are sometimes unnoticeable even on the charts.

**Conclusion.** The results obtained in the work give a possibility to build stochastic models of diffusion processes with semi-Markov switches.

The asymptotic properties of the stochastic approximation procedure make it possible to establish the equilibrium points of the original system. We believe that this type of modeling allows us to establish equilibrium points for models of the spread of epidemics.

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## МОДЕЛЬ СПОЖИВАННЯ АЛКОГОЛЮ З НАПІВМАРКІВСЬКИМИ КОЕФІЦІЄНТАМИ

Стаття присвячена застосуванню неперервної процедури стохастичної апроксимації, що включає напівмарківські перемикання в схемі усереднення з використанням малого параметру. Ми представляємо модель, у якій на об'єкти впливають шуми, що залежать від напівмарківського процесу. Акцент робиться на аналіз збіжності та стійкості процесу стохастичного наближення в цьому контексті. Теоретичні результати представлені разом із чисельним моделюванням, що демонструють практичне застосування в управлінні складними стохастичними системами. Ця робота відкриває способи для використання підходів стохастичного наближення в ширшій області напівмарківських процесів. Визнаючи мінливий характер споживання алкоголю та його залежність від різноманітних факторів, ми пропонуємо закодувати випадкову динаміку у напівмарківській структурі параметрів. Розглянута модель описує індивідуальне споживання алкоголю як напівмарківський процес; ймовірності переходу між різними станами споживання алкоголю визначаються соціально-демографічними змінними, які змінюються з часом. Таким чином, наш підхід долає розрив між реаліями тенденцій споживання алкоголю, що постійно змінюються, і статичними традиційними моделями зі сталими коефіцієнтами. Інтегруючи змінні реального світу в нашу інноваційну модель, ми пропонуємо новітній аналі-

тичний інструмент, який прокладає нові шляхи для розуміння та вирішення проблем, пов'язаних із моделями вживання алкоголю. Крім того, наші висновки можуть суттєво вплинути на формулювання ефективних стратегій і заходів у сфері охорони здоров'я, а також спрямованих на зменшення шкоди, пов'язаної з алкоголем.

**Ключові слова:** *напівмарковські процеси, дифузійні процеси, вінерівські процеси, стохастичні моделі, стохастичні процеси.*

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# ІНТЕРПОЛЯЦІЙНА ЗАДАЧА У ВАГОВИХ ПРОСТОРАХ ПЕЛІ-ВІНЕРА

Ю. Любарський і К. Сейп (Revista Matematica Iberoamericana, 1997 (13), № 2) дослідили критерій існування єдиного розв'язку простої інтерполяційної задачі $f(\lambda_k) = b_k$ в термінах умов Макенхоупта (неперервної та дискретної ($A^p$) умови) у просторах Пелі-Вінера цілих функцій експоненційного типу, що не перевищує $\pi$, чиє звуження на дійсну вісь співпадає з простором функцій, степінь порядку $p$ модуля яких є інтегрованим за Лебегом на цій осі, з $p$-нормою (тут $p$ є дійсним числом, більшим за 1). Ці результати дають можливість отримати критерій безумовної базисності системи експонент в просторі функцій, степінь порядку $p$ модуля яких є інтегрованою за Лебегом на $(-\pi; \pi)$ функцією. При цьому послідовність комплексних чисел $(\lambda_k)$ з єдиною граничною точкою на нескінченності, для якої згадана інтерполяційна задача має єдиний розв'язок, називається повною інтерполяційною послідовністю в згаданому просторі Пелі-Вінера.

Згадані результати були узагальненням для випадку $p = 2$ результатів Павлова (1979), Нікольського (1980) та Мінкіна (1982). Ми ж узагальнюємо ці результати на вагові простори (вагою є степенева функція з показником степеня $\omega$) цілих функцій експоненційного типу, що не перевищує $\sigma$, де $\sigma$ – невід'ємне дійсне число, $\omega$ – дійсне число, більше від –1, з $p$-нормою, тобто знаходимо умови повноти послідовності інтерполяційної послідовності $(\lambda_k)$ у ваговому просторі Пелі-Вінера. Розглядаємо різні форми цих умов, серед яких і умови Макенхоупта, неперервна та дискретна ($A^p$) умови. Доведено також, що якщо послідовність комплексних чисел є повною інтерполяційною послі-

 