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SMOOTHNESS EFFECTS ON NUMERICAL INTEGRATION ACCURACY FOR RAPIDLY OSCILLATING BIVARIATE FUNCTIONS ON SPARSE GRIDS

One of the key tasks in modern applied mathematics, without which it is impossible to model and analyze complex processes, particularly in digital image processing, is the numerical integration of functions of several variables. The main problem of numerical integration of functions of several variables is the increase in computational costs with increasing dimension of the integration domain.

Of particular interest are numerical integration methods developed using information operators that restore intermediate values of quantities based on a given set of known values of a function of several variables at points, lines, planes, etc. Based on such operators, economical schemes for interpolating functions of several variables are constructed. The use of economical schemes in the numerical integration of functions of two and three variables allows one to construct sparse grids and calculate approximate integrals with less data and with a predetermined accuracy compared to classical methods.

The purpose of this article is to demonstrate the use of economical interpolation schemes for approximate calculation of double integrals of rapidly oscillating functions of general form on different classes of smoothness. The paper analyzes the influence of the order of differentiability of a function on the rate of decay of the theoretical error of approximation of cubature formulas. It is shown that as the smoothness of the function increases, the estimates of the error of numerical integration improve, which allows the effective use of sparse grids without loss of accuracy. The results obtained establish a quantitative relationship between the class of differentiability of a function, the discretization parameters, and the frequency of oscillations, and can be used to justify the choice of numerical methods for integrating rapidly oscillating functions of two variables.

Key words: *mathematical modeling of processes, digital image processing, numerical integration, rapidly oscillating functions of many variables, cubature formula, function interpolation, sparse grids.*

Introduction. In modern applied mathematics, numerical integration of functions of several variables is one of the important tasks [1-5]. In mathematical physics, such tasks arise when modeling heat conduction, diffusion, distribution of potentials, energies, and other physical quantities in continuous media, and in digital signal and image processing tasks, the calculation of multidimensional integrals arises in filtering, signal reconstruction, spectral analysis, and the calculation of correlation and statistical characteristics. It is important to note that as the dimension of the integration region increases, the computational complexity increases, so it is important to find effective methods for numerical integration of functions of several variables. Currently, there are methods of numerical integration, including methods of numerical integration of rapidly oscillating functions of several variables, which are developed using information operators that restore intermediate values of functions based on known values of the function at points, lines, planes, etc. [6-11]. We refer to such information operators as those developed by O. M. Lytvyn [12-14], on the basis of which economical schemes for interpolating functions of several variables have been created. The use of economical interpolation schemes in constructing cubic formulas for the approximate calculation of rapidly oscillating functions of several general types made it possible to perform calculations with less data compared to classical methods [15]. This article discusses the use of economical interpolation schemes for approximate calculation of double integrals of rapidly oscillating functions of general form for higher orders of differentiability. The main focus is on how the error estimate of numerical integration improves with increasing smoothness of the function, which allows for the effective use of a sparse grid.

Problem statement. Consider $H^{2,r}\left(M,\tilde{M}\right)$ – the class of real functions defined $G=\left[0,1\right]^2$ and such that

$$\left|f^{(r,0)}\left(x,y\right)\right|\le M,\quad \left|f^{(0,r)}\left(x,y\right)\right|\le M,\quad \left|f^{(r,r)}\left(x,y\right)\right|\le \tilde{M},\quad r=1,2\,.$$

For approximate calculation of the integral of functions of two variables of the form

$$\tilde{I}\left(\omega\right)=\int_0^1\int_0^1 e^{i\omega g(x,y)}dxdy\,,\ \omega\gg 2\pi \qquad (1)$$

construct a cubature formula using linear spline interpolation operators based on linear spline O. M. Lytvyn's operators on the class $H^{2,2}\left(M,\tilde{M}\right)$. Compare the accuracy of the cubature formula on the class $H^{2,1}\left(M,\tilde{M}\right)$ and $H^{2,2}\left(M,\tilde{M}\right)$.

Numerical integration of rapidly oscillating functions of general form on a class $H^{2,r}\left(M,\tilde{M}\right)$, $r=1,2$.

Let us introduce the following notation:

$$h1_{10}(x)=\begin{cases}0, & x\le x_0,\\ \dfrac{x-x_1}{-\Delta_1}, & x_0<x<x_1,\\ 0, & x\ge x_1,\end{cases}\quad H1_{10}(y)=\begin{cases}0, & y\le y_0,\\ \dfrac{y-y_1}{-\Delta_1}, & y_0<y<y_1,\\ 0, & y\ge \tilde{y}_1,\end{cases}$$

$$h1_{1k}(x)=\begin{cases}0, & x\le x_{k-1},\\ \dfrac{x-x_{k-1}}{\Delta_1}, & x_{k-1}<x<x_k,\\ \dfrac{x-x_{k+1}}{-\Delta_1}, & x_k\le x<x_{k+1},\\ 0, & x\ge x_{k+1},\end{cases}\quad k=\overline{1,\ell_1-1},$$

$$H1_{1j}(y)=\begin{cases}0, & y\le y_{j-1},\\ \dfrac{y-y_{j-1}}{\Delta_1}, & y_{j-1}<y<y_j,\\ \dfrac{y-y_{j+1}}{-\Delta_1}, & y_j\le y<y_{j+1},\\ 0, & y\ge y_{j+1}\end{cases}\quad j=\overline{1,\ell_1-1},$$

$$h1_{1\ell_1}(x)=\begin{cases}0, & x\le x_{\ell_1-1},\\ \dfrac{x-x_{\ell_1}}{\Delta_1}, & x_{\ell_1-1}<x<x_{\ell_1},\\ 0, & x\ge x_{\ell_1},\end{cases}\quad H1_{1\ell_1}(y)=\begin{cases}0, & y\le y_{\ell_1-1},\\ \dfrac{y-y_{\ell_1}}{\Delta_1}, & y_{\ell_1-1}<y<y_{\ell_1},\\ 0, & y\ge y_{\ell_1},\end{cases}$$

$$x_k=k\Delta_1,\ y_j=j\Delta_1,\ \Delta_1=\frac{1}{\ell_1};$$

$$\tilde{h}1_{10}(x)=\begin{cases}0, & x\le \tilde{x}_0,\\ \dfrac{x-\tilde{x}_1}{-\Delta_1}, & \tilde{x}_0<x<\tilde{x}_1,\\ 0, & x\ge \tilde{x}_1,\end{cases}\quad \tilde{H}1_{10}(y)=\begin{cases}0, & y\le \tilde{y}_0,\\ \dfrac{y-\tilde{y}_1}{-\Delta_1}, & \tilde{y}_0<y<\tilde{y}_1,\\ 0, & \tilde{y}\ge y_1,\end{cases}$$

$$\tilde{h}1_{1k}(x)=\begin{cases}0, & x\le \tilde{x}_{\tilde{k}-1},\\ \dfrac{x-\tilde{x}_{\tilde{k}-1}}{\Delta_1}, & \tilde{x}_{\tilde{k}-1}<x<\tilde{x}_{\tilde{k}}\\ \dfrac{x-\tilde{x}_{\tilde{k}+1}}{-\Delta_1}, & \tilde{x}_{\tilde{k}}\le x<\tilde{x}_{\tilde{k}+1},\\ 0, & x\ge \tilde{x}_{\tilde{k}+1},\end{cases}\quad \tilde{k}=\overline{1,\ell_1^{\,2}-1},$$

$$\tilde{H}1_{1\tilde{j}}(y)=\begin{cases}0, & y\le \tilde{y}_{\tilde{j}-1},\\ \dfrac{y-\tilde{y}_{\tilde{j}-1}}{\Delta_1}, & \tilde{y}_{\tilde{j}-1}\le y<\tilde{y}_{\tilde{j}},\\ \dfrac{y-y_{\tilde{j}+1}}{-\Delta_1}, & \tilde{y}_{\tilde{j}}\le y<\tilde{y}_{\tilde{j}+1},\\ 0, & y\ge \tilde{y}_{\tilde{j}+1},\end{cases}\qquad \tilde{j}=\overline{1,\ell_1^{\,2}-1},$$

$$\tilde{h}1_{1\ell_1^2}(x)=\begin{cases}0, & x\le \tilde{x}_{\ell_1^2-1},\\ \dfrac{x-\tilde{x}_{\ell_1^2}}{\Delta_1}, & \tilde{x}_{\ell_1^2-1}\le x<\tilde{x}_{\ell_1^2},\\ 0, & x\ge \tilde{x}_{\ell_1^2},\end{cases}$$

$$\tilde{H}1_{1\ell_1^2}(y)=\begin{cases}0, & y\le \tilde{y}_{\ell^2-1},\\ \dfrac{y-\tilde{y}_{\ell_1^2}}{\Delta_1}, & \tilde{y}_{\ell_1^2-1}\le y<\tilde{y}_{\ell_1^2},\\ 0, & y\ge \tilde{y}_{\ell_1^2},\end{cases}$$

$$\tilde{x}_{\tilde{k}}=\tilde{k}\Delta_1,\ \tilde{y}_{\tilde{j}}=\tilde{j}\Delta_1,\ \tilde{k},\tilde{j}=\overline{0,\ell_1^{\,2}},\ \Delta_1=\frac{1}{\ell_1^{\,2}}.$$

Auxiliary functions

$h2_{10}(x), H2_{10}(y)$, $h2_{1p}(x),\ p=\overline{1,\ell_2-1},\ \ H2_{1s}(y),\ s=\overline{1,\ell_2-1}$,

$h2_{1\ell_2}(x), H2_{1\ell_2}(y)$, $\tilde{h}2_{10}(x), \tilde{H}2_{10}(y),\ \ \tilde{h}2_{1\tilde{p}}(x),\ \ \tilde{p}=\overline{1,\ell_2^{\,2}-1},$

$\tilde{H}2_{1\tilde{s}}(y),\ \ \tilde{s}=\overline{1,\ell_2^{\,2}-1},\ \tilde{h}2_{1\ell_2^2}(x),\ \tilde{H}2_{1\ell_2^2}(y)$

are defined similarly to

$\tilde{x}_p=\tilde{p}\tilde{\Delta}_2-\tilde{\Delta}_2/2,\ \ \tilde{y}_s=\tilde{s}\tilde{\Delta}_2-\tilde{\Delta}_2/2,\ \ \tilde{p},\tilde{s}=\overline{1,\ell_2^{\,2}}$, $\tilde{\Delta}_2=1/\ell_2^{\,2}$.

Let's consider operators

$$Jf(x,y)=\sum_{k=0}^{\ell_1} f(x_k,y)h1_{1k}(x)+\sum_{j=0}^{\ell_1} f(x,y_j)H1_{1j}(y)-$$

$$-\sum_{k=0}^{\ell_1}\sum_{j=0}^{\ell_1} f(x_k,y_j)h1_{1k}(x)H1_{1j}(y);$$

$$\tilde{J}f(x,y)=\sum_{k=0}^{\ell_1}\sum_{\tilde{j}=0}^{\ell_1^2} f(x_k,\tilde{y}_{\tilde{j}})h1_{1k}(x)\tilde{H}1_{1\tilde{j}}(y)+$$

$$+\sum_{j=0}^{\ell_1}\sum_{\tilde{k}=0}^{\ell_1^2} f\left(\tilde{x}_{\tilde{k}}, y_j\right)\tilde{h}1_{1\tilde{k}}(x)H1_{1j}(y)-$$

$$-\sum_{k=0}^{\ell_1}\sum_{j=0}^{\ell_1} f\left(x_k, y_j\right)h1_{1k}(x)H1_{1j}(y);$$

$$Og(x, y)=\sum_{p=0}^{\ell_2} g\left(x_p, y\right)h2_{1p}(x)+\sum_{s=0}^{\ell_2} g(x, y_s)H2_{1s}(y)-$$

$$-\sum_{p=0}^{\ell_2}\sum_{s=0}^{\ell_2} g\left(x_p, y_s\right)h2_{1p}(x)H2_{1s}(y);$$

$$\tilde{O}g(x, y)=\sum_{p=0}^{\ell_2}\sum_{\tilde{s}=0}^{\ell_2^2} g\left(x_p, \tilde{y}_{\tilde{s}}\right)h2_{1p}(x)\tilde{H}2_{1\tilde{s}}(y)+$$

$$+\sum_{s=0}^{\ell_2}\sum_{\tilde{p}=0}^{\ell_2^2} g\left(\tilde{x}_{\tilde{p}}, y_j\right)\tilde{h}2_{1\tilde{p}}(x)H2_{1s}(y)-\sum_{p=0}^{\ell_2}\sum_{s=0}^{\ell_2} g\left(x_p, y_s\right)h2_{1p}(x)H2_{1s}(y).$$

If we introduce additional operators

$$J_1 f(x, y)=\sum_{k=0}^{\ell_1} f(x_k, y)h1_{1k}(x), \quad J_2 f(x, y)=\sum_{j=0}^{\ell_1} f\left(x, y_j\right)H1_{1j}(y);$$

$$\tilde{J}_1 f(x, y)=\sum_{\tilde{k}=0}^{\ell_1^2} f\left(\tilde{x}_{\tilde{k}}, y\right)\tilde{h}1_{1\tilde{k}}(x), \quad \tilde{J}_2 f(x, y)=\sum_{\tilde{j}=0}^{\ell_1^2} f\left(x, \tilde{y}_{\tilde{j}}\right)\tilde{H}1_{1\tilde{j}}(y);$$

$$O_1 g(x, y)=\sum_{p=0}^{\ell_2} g\left(x_p, y\right)h2_{1p}(x), \quad O_2 g(x, y)=\sum_{s=0}^{\ell_2} g(x, y_s)H2_{1s}(y);$$

$$\tilde{O}_1 g(x, y)=\sum_{\tilde{p}=0}^{\ell_2^2} g\left(\tilde{x}_{\tilde{p}}, y_j\right)\tilde{h}2_{1\tilde{p}}(x), \quad \tilde{O}_2 g(x, y)=\sum_{\tilde{s}=0}^{\ell_2^2} g\left(x_p, \tilde{y}_{\tilde{s}}\right)\tilde{H}2_{1\tilde{s}}(y),$$

then for interlinear operators $Jf(x, y)$, $Og(x, y)$ and interpolating operators $\tilde{J}f(x, y)$, $\tilde{O}g(x, y)$, the following identities hold:

$$Jf=\left(J_1+J_2-J_1J_2\right)f, \quad \tilde{J}f=\left(J_1\tilde{J}_2+\tilde{J}_1J_2-J_1J_2\right)f;$$

$$Og=\left(O_1+O_2-O_1O_2\right)g, \quad \tilde{O}g=\left(O_1\tilde{O}_2+\tilde{O}_1O_2-O_1O_2\right)g.$$

The following cubature formula

$$\Phi=\int_0^1\int_0^1 \tilde{J}f(x, y)\,dxdy$$

is proposed for the approximate calculation of the integral

$$I = \int_0^1 \int_0^1 f(x, y)\,dxdy \,.$$

Theorem 1. *[15] Let* $f(x, y) \in H^{2,1}(M, \tilde{M})$, *then*

$$\rho(I, \Phi) = \left| \int_0^1 \int_0^1 f(x, y)\,dxdy - \int_0^1 \int_0^1 \tilde{J}f(x, y)\,dxdy \right| \le \frac{\tilde{M} + 6M}{9\ell_1^2} .$$

Theorem 2. *Let* $f(x, y) \in H^{2,2}(M, \tilde{M})$, *then*

$$\rho(I, \Phi) = \left| \int_0^1 \int_0^1 f(x, y)\,dxdy - \int_0^1 \int_0^1 \tilde{J}f(x, y)\,dxdy \right| \le \frac{\tilde{M} + 24M}{144\ell_1^4} .$$

Proof. Let us consider additional functions that will be used in proving the theorem for representing the approximation error of $f(x, y)$ by the interlinear operator $Jf(x, y)$ through $f^{(2,2)}(x, y)$:

$$K_{1k}(x,\xi) = \begin{cases} \dfrac{x_{k+1} - x}{x_{k+1} - x_k}(x_k - \xi), & x_k \le \xi < x, \\ \dfrac{x_k - x}{x_{k+1} - x_k}(x_{k+1} - \xi), & x < \xi \le x_{k+1}, \end{cases}$$

$$K_{2j}(y,\eta) = \begin{cases} \dfrac{y_{j+1} - y}{y_{j+1} - y_j}(y_j - \eta), & y_j \le \eta < y, \\ \dfrac{y_j - y}{y_{j+1} - y_j}(y_{j+1} - \eta), & y < \eta \le y_{j+1}, \end{cases}$$

$$\tilde{K}_{1\tilde{k}}(x,\tilde{\xi}) = \begin{cases} \dfrac{\tilde{x}_{\tilde{k}+1} - x}{\tilde{x}_{\tilde{k}+1} - \tilde{x}_{\tilde{k}}}(\tilde{x}_{\tilde{k}} - \tilde{\xi}), & \tilde{x}_{\tilde{k}} \le \tilde{\xi} < x, \\ \dfrac{\tilde{x}_{\tilde{k}} - x}{\tilde{x}_{\tilde{k}+1} - \tilde{x}_{\tilde{k}}}(\tilde{x}_{\tilde{k}+1} - \tilde{\xi}), & x < \tilde{\xi} \le \tilde{x}_{\tilde{k}+1}, \end{cases}$$

$$\tilde{K}_{2\tilde{j}}(y,\tilde{\eta}) = \begin{cases} \dfrac{\tilde{y}_{\tilde{j}+1} - y}{\tilde{y}_{\tilde{j}+1} - \tilde{y}_{\tilde{j}}}(\tilde{y}_{\tilde{j}} - \tilde{\eta}), & \tilde{y}_{\tilde{j}} \le \tilde{\eta} < y, \\ \dfrac{\tilde{y}_{\tilde{j}} - y}{\tilde{y}_{\tilde{j}+1} - \tilde{y}_{\tilde{j}}}(\tilde{y}_{\tilde{j}+1} - \tilde{\eta}), & y < \tilde{\eta} \le \tilde{y}_{\tilde{j}+1}. \end{cases}$$

Let's find the estimate $\rho(I, \Phi)$:

$$\rho\left(I,\Phi\right)=\left|\int_0^1\int_0^1 f(x,y)dxdy-\int_0^1\int_0^1\tilde{J}f(x,y)dxdy\right|\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)+Jf(x,y)-\tilde{J}f(x,y)\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)\right|dxdy+\int_0^1\int_0^1\left|Jf(x,y)-\tilde{J}f(x,y)\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)\right|dxdy+$$

$$+\int_0^1\int_0^1\left|\left(J_1+J_2-J_1J_2\right)f(x,y)-\left(J_1\tilde{J}_2+J_2\tilde{J}_1-J_1J_2\right)f(x,y)\right|dxdy\le$$

$$\le\sum_{k=0}^{\ell_1-1}\sum_{j=0}^{\ell_1-1}\int_{x_k}^{x_{k+1}}\int_{y_j}^{y_{j+1}}\left|\int_{x_k}^{x_{k+1}}\int_{y_j}^{y_{j+1}}f^{(2,2)}\left(\xi,\eta\right)K_{1k}\left(x,\xi\right)K_{2j}\left(y,\eta\right)d\xi d\eta\right|dxdy+$$

$$+\sum_{j=0}^{\ell_1-1}\sum_{\tilde{k}=0}^{\ell_1^2-1}\int_{\tilde{x}_{\tilde{k}}}^{\tilde{x}_{\tilde{k}+1}}\int_{\tilde{x}_k}^{\tilde{x}_{k+1}}\left|f^{(2,0)}(\tilde{\xi},y_j)\right|\left|\tilde{K}_{1\tilde{k}}(x,\tilde{\xi})\right|d\tilde{\xi}dx\int_{y_j}^{y_{j+1}}dy+$$

$$+\sum_{k=0}^{\ell_1-1}\sum_{\tilde{j}=0}^{\ell_1^2-1}\int_{x_k}^{x_{k+1}}dx\int_{\tilde{y}_{\tilde{j}}}^{\tilde{y}_{\tilde{j}+1}}\int_{y_j}^{y_{j+1}}\left|f^{(0,2)}(x_k,\tilde{\eta})\right|\left|\tilde{K}_{2_{\tilde{j}}}(y,\tilde{\eta})\right|d\tilde{\eta}dy\,.$$

Note that

$$\int_{x_k}^{x_{k+1}}\left|K_{1k}\left(x,\xi\right)\right|d\xi=\frac{x_{k+1}-x}{x_{k+1}-x_k}\int_{x_k}^{x}\left|x_k-\xi\right|d\xi+\frac{x-x_k}{x_{k+1}-x_k}\int_{x}^{x_{k+1}}\left|x_{k+1}-\xi\right|d\xi=$$

$$=\frac{x_{k+1}-x}{x_{k+1}-x_k}\left(x-x_k\right)+\frac{x-x_k}{x_{k+1}-x_k}\left(x_{k+1}-x\right),$$

and

$$\int_{x_k}^{x_{k+1}}\int_{x_k}^{x_{k+1}}\left|K_{1k}\left(x,\xi\right)\right|d\xi dx=$$

$$=\int_{x_k}^{x_{k+1}}\left(\frac{x_{k+1}-x}{x_{k+1}-x_k}\left(x-x_k\right)+\frac{x-x_k}{x_{k+1}-x_k}\left(x_{k+1}-x\right)\right)dx=$$

$$=\frac{1}{2\Delta}\int_{x_k}^{x_{k+1}}\frac{\left(x-x_k\right)^3}{3}dx+\frac{1}{2\Delta_1}\int_{x_k}^{x_{k+1}}\frac{\left(x_{k+1}-x\right)^3}{3}dx=$$

$$= \frac{1}{2\Delta_1}\left(\frac{(x-x_k)^4}{4\cdot 3}\Bigg|_{x_k}^{x_{k+1}} - \frac{(x_{k+1}-x)^4}{4\cdot 3}\Bigg|_{x_k}^{x_{k+1}} \right) = \frac{1}{2\Delta_1}\left(\frac{\Delta_1^{\,4}}{3\cdot 4} + \frac{\Delta_1^{\,4}}{3\cdot 4} \right) = \frac{2\Delta_1^{\,3}}{4!} = \frac{\Delta_1^{\,3}}{12} .$$

So,

$$\rho(I,\Phi) \le \tilde{M}\,\frac{\Delta_1^{\,2}}{12}\,\frac{\Delta_1^{\,2}}{12} + \;2M\,\ell_1\Delta_1\,\frac{2\tilde{\Delta}_1^{3}}{4!}\,\ell_1^{\,2} =$$

$$= \frac{\tilde{M}}{144\ell_1^{\,4}} + \frac{M}{6\ell_1^{\,4}} = \frac{\tilde{M}+24M}{144\ell_1^{\,4}} .$$

Theorem 2 is proven.

The following cubature formula

$$\tilde{\Phi}(\omega) = \int_0^1\int_0^1 \mathrm{e}^{i\omega\tilde{O}g(\mathrm{x,y})}\,dxdy \tag{2}$$

is proposed for the approximate calculation of integral (1).

Theorem 3. *[15] Let* $g(x,y)\in H^{2,1}\left(M,\tilde{M}\right)$, *then*

$$\rho\left(\tilde{I}(\omega),\tilde{\Phi}(\omega)\right) = \left| \int_0^1\int_0^1 e^{i\omega g(x,y)}dxdy - \int_0^1\int_0^1 e^{i\omega\tilde{O}g(x,y)}dxdy \right| \le \min\left(4;\ \frac{\omega\left(\tilde{M}+6M\right)}{9\ell_2^{\,2}}\right).$$

Theorem 4. *Let* $f(x,y),\ g(x,y)\in H^{2,2}\left(M,\tilde{M}\right)$, *then*

$$\rho\left(\tilde{I}(\omega),\tilde{\Phi}(\omega)\right) = \left| \int_0^1\int_0^1 e^{i\omega g(x,y)}dxdy - \int_0^1\int_0^1 e^{i\omega Og(x,y)}dxdy \right| \le \min\left(4;\ \frac{\omega\left(\tilde{M}+24M\right)}{144\ell_2^{\,4}}\right).$$

Proof. Let us consider additional functions that will be used in proving the theorem for representing the approximation error of $g(x,y)$ by the interlinear operator $Og(x,y)$ through $g^{(2,2)}(x,y)$:

$$G_{1p}(x,t) = \begin{cases} \dfrac{x_{p+1}-x}{x_{p+1}-x_p}\left(x_p - t\right),\ x_p \le t < x, \\[2ex] \dfrac{x_p - x}{x_{p+1}-x_p}\left(x_{p+1}-t\right),\ \ x < t \le x_{p+1}, \end{cases}$$

$$G_{2,s}(y,\tau) = \begin{cases} \dfrac{y_{s+1}-y}{y_{s+1}-y}\left(y_s-\tau\right),\ y_s \le \tau < y, \\[2ex] \dfrac{y_s-y}{y_{s+1}-y}\left(y_{s+1}-\tau\right),\ y < \tau \le y_{s+1}. \end{cases}$$

$$\tilde{G}_{1p}\left(x,\tilde{t}\right)=\begin{cases}\dfrac{\tilde{x}_{\tilde{p}+1}-x}{\tilde{x}_{\tilde{p}+1}-\tilde{x}_{\tilde{p}}}\left(\tilde{x}_{\tilde{p}}-t\right),\ \tilde{x}_{\tilde{p}}\le t<x,\\ \dfrac{\tilde{x}_{\tilde{p}}-x}{\tilde{x}_{\tilde{p}+1}-\tilde{x}_{\tilde{p}}}\left(\tilde{x}_{\tilde{p}+1}-t\right),\ \ x<t\le \tilde{x}_{\tilde{p}+1},\end{cases}$$

$$\tilde{G}_{2,\tilde{s}}\left(y,\tau\right)=\begin{cases}\dfrac{\tilde{y}_{\tilde{s}+1}-y}{\tilde{y}_{\tilde{s}+1}-\tilde{y}_{\tilde{s}}}\left(\tilde{y}_{\tilde{s}}-\tau\right),\ \tilde{y}_{\tilde{s}}\le \tau<y,\\ \dfrac{\tilde{y}_{\tilde{s}}-y}{\tilde{y}_{\tilde{s}+1}-\tilde{y}_{\tilde{s}}}\left(\tilde{y}_{\tilde{s}+1}-\tau\right),\ y<\tau\le \tilde{y}_{\tilde{s}+1}.\end{cases}$$

Let's find the estimate $\rho\left(\tilde{I}(\omega),\tilde{\Phi}\left(\omega\right)\right)$:

$$\rho\left(\tilde{I}(\omega),\tilde{\Phi}\left(\omega\right)\right)=\left|\int_0^1\int_0^1 e^{i\omega g(x,y)}dxdy-\int_0^1\int_0^1 e^{i\omega\tilde{O}g(x,y)}dxdy\right|\le$$

$$\le\left|\int_0^1\int_0^1 e^{i\omega g(x,y)}dxdy-\int_0^1\int_0^1 e^{i\omega Og(x,y)}dxdy\right|+$$

$$+\left|\int_0^1\int_0^1 e^{i\omega Og(x,y)}dxdy-\int_0^1\int_0^1 e^{i\omega\tilde{O}g(x,y)}dxdy\right|\le$$

$$\le\int_0^1\int_0^1\left|e^{i\omega g(x,y)}-e^{i\omega Og(x,y)}\right|dxdy+\int_0^1\int_0^1\left|e^{i\omega Og(x,y)}-e^{i\omega\tilde{O}g(x,y)}\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|2i\sin\frac{\omega g(x,y)-\omega Og(x,y)}{2}e^{i\frac{\omega}{2}\left(g(x,y)+Og(x,y)\right)}\right|dxdy+$$

$$+\int_0^1\int_0^1\left|2i\sin\frac{\omega Og(x,y)-\omega\tilde{O}g(x,y)}{2}e^{i\frac{\omega}{2}\left(Og(x,y)+\tilde{O}g(x,y)\right)}\right|dxdy\le$$

$$\le 2\int_0^1\int_0^1\left|\sin\frac{\omega g(x,y)-\omega Og(x,y)}{2}\right|dxdy+$$

$$+2\int_0^1\int_0^1\left|\sin\frac{\omega Og(x,y)-\omega\tilde{O}g(x,y)}{2}\right|dxdy\le$$

$$\le 2\int_0^1\int_0^1\min\left(1;\frac{\omega\left|g\left(x,y\right)-Og\left(x,y\right)\right|}{2}\right)dxdy+$$

$$+2\int_0^1\int_0^1 \min\left(1;\frac{\omega\left|\left(O_1+O_2-O_1O_2\right)g(x,y)-\left(O_1\tilde{O}_2+O_2\tilde{O}_1-O_1O_2\right)g(x,y)\right|}{2}\right)dxdy\le$$

$$\le 2\int_0^1\int_0^1 \min\left(1;\frac{\omega\left|g(x,y)-Og(x,y)\right|}{2}\right)dxdy+$$

$$+2\int_0^1\int_0^1 \min\left(1;\frac{\omega\left(O_1-O_1\tilde{O}_2\right)g(x,y)+\left(O_2-O_2\tilde{O}_1\right)g(x,y)}{2}\right)dxdy\le$$

$$\le 2\sum_{p=0}^{\ell_2-1}\sum_{s=0}^{\ell_2-1}\int_{x_p}^{x_{p+1}}\int_{y_s}^{y_{s+1}}\min\left(1;\frac{\omega}{2}\left|\int_{x_p}^{x_{p+1}}\int_{y_s}^{y_{s+1}} g^{(2,2)}(t,\tau)G_{1p}(x,t)G_{2s}(y,\tau)\,dtd\eta\right|\right)dxdy+$$

$$+2\min\left(\sum_{\tilde{p}=0}^{\ell_2^2-1}\sum_{\tilde{s}=0}^{\ell_2^2-1}\int_{\tilde{x}_{\tilde{p}}}^{\tilde{x}_{\tilde{p}+1}}\int_{\tilde{y}_{\tilde{s}}}^{\tilde{y}_{\tilde{s}+1}} dxdy;\ \frac{\omega}{2}\sum_{p=0}^{\ell_2-1}\sum_{\tilde{s}=0}^{\ell_2^2-1}\int_{x_p}^{x_{p+1}} dx\int_{\tilde{y}_{\tilde{s}}}^{\tilde{y}_{\tilde{s}+1}}\int_{y_j}^{\tilde{y}_{j+1}}\left|g^{(0,2)}(x_p,y)\right|\left|G_{2_{\tilde{s}}}(y,\tilde{\tau})\right|d\tilde{\tau}dy+\right.$$

$$\left.+\frac{\omega}{2}\sum_{s=0}^{\ell_2-1}\sum_{\tilde{p}=0}^{\ell_2^2-1}\int_{x_p}^{x_{p+1}}\int_{\tilde{x}_p}^{\tilde{x}_{p+1}}\left|g^{(2,0)}(\tilde{t},y_s)\right|\left|\tilde{G}_{1\tilde{p}}(x,\tilde{t})\right|d\tilde{t}dx\int_{y_s}^{y_{s+1}} dy\right)\le$$

$$\le 2\min\left(\ell_2^2\Delta_2^2,\ \frac{\tilde{M}\omega}{2}\frac{\Delta_2^2}{12}\frac{\Delta_2^2}{12}\right)+$$

$$+2\min\left(\left(\ell_2^2\Delta_2^2\right)^2,\ \frac{2M\omega}{2}\ell_2\Delta_2\ell_2^2\frac{2\tilde{\Delta}_2^3}{4!}\right)=$$

$$=\min\left(2;\frac{M\omega}{144\ell_2^4}\right)+\min\left(2;\frac{M\omega}{6\ell_2^4}\right)=\min\left(4;\frac{\omega\left(\tilde{M}+24M\right)}{144\ell_2^4}\right).$$

Theorem 4 is proven.

The results obtained allow us to construct estimates of approximation errors for integrals of rapidly oscillating functions of a more general form, in particular for the integral

$$I^2(\omega)=\int_0^1\int_0^1 f(x,y)e^{i\omega g(x,y)}dxdy\,. \tag{3}$$

Comparison of the accuracy of the cubature formula on classes $H^{2,1}\left(M,\tilde{M}\right)$ and $H^{2,2}\left(M,\tilde{M}\right)$. The efficiency of numerical integration of functions of two variables significantly depends on the smoothness class of the integrand. For rapidly oscillating integrals, the order of differentiability determines the possibility of obtaining more accurate theoretical estimates of error and justifies the choice of discretization parameters. In this regard, it is advisable to compare the accuracy of quadrature formulas on different classes of functions that differ in their level of smoothness.

Consider the function $\sin(x+y)$, which belongs to the class $H^{2,r}\left(M,\tilde{M}\right)$, $r=1,2$, $M=\tilde{M}=1$.

For a given function, consider the behavior of the error of the cubature formula (2). Pay particular attention to analyzing the dependence of the rate of error decay on the order of differentiability and the influence of this factor on the efficiency of using sparse grids.

According to Theorem 2, the accuracy of the approximation of integral (1) by formula (2) on the class $H^{2,1}\left(M,\tilde{M}\right)$ is equal to $\varepsilon_1=\varepsilon_1(\omega)=\dfrac{7\omega}{9\ell^2}$, when $\ell_2=\ell$, and according to Theorem 4 – on the class $H^{2,2}\left(M,\tilde{M}\right)$ it is equal to $\varepsilon_2=\varepsilon_2(\omega)=\dfrac{25\omega}{144\ell^4}$, when $\ell_2=\ell$.

Table 1 shows the values of theoretical errors ε_1 and ε_2 in class $H^{2,1}\left(M,\tilde{M}\right)$, $H^{2,2}\left(M,\tilde{M}\right)$ respectively, and for different values of ω and ℓ.

Table 1

Comparison of theoretical errors at $\omega=20\pi, 60\pi, 100\pi$

ℓ	$\omega=20\pi$	ε_1	ε_2
64	$\omega=20\pi$	1,193e -02	6,502e-07
128	$\omega=20\pi$	2,983e-03	4,064e-08
256	$\omega=20\pi$	7,456e-04	2,539e-10
512	$\omega=20\pi$	1,864e-04	1,587e-10
1024	$\omega=20\pi$	4,660e-05	9,921e-12
256	$\omega=60\pi$	2,273e-03	7,619e-09
512	$\omega=60\pi$	5,592 e-04	4,762e-10
1024	$\omega=60\pi$	1,398e-04	2,976e-11
512	$\omega=100\pi$	9,321e-04	7,936e-10
1024	$\omega=100\pi$	2,330e-04	4,960e-11

Fig. 1 and Fig. 2 show the curves of theoretical trajectories ε_1 and ε_2 for different values of ω and ℓ.

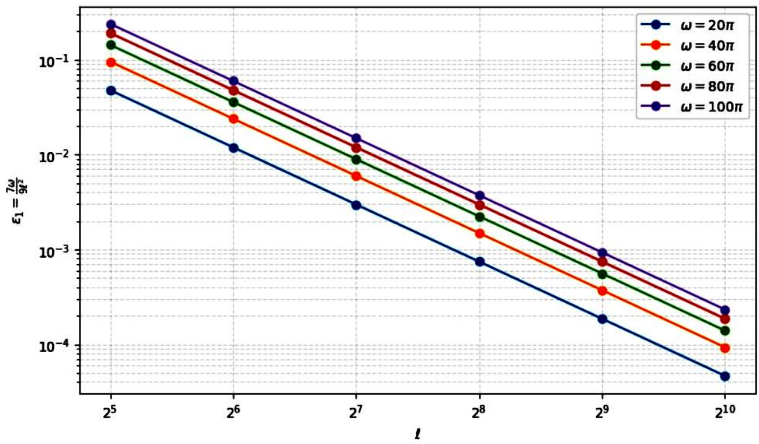

Fig. 1. *Visualization of error values* ε_1

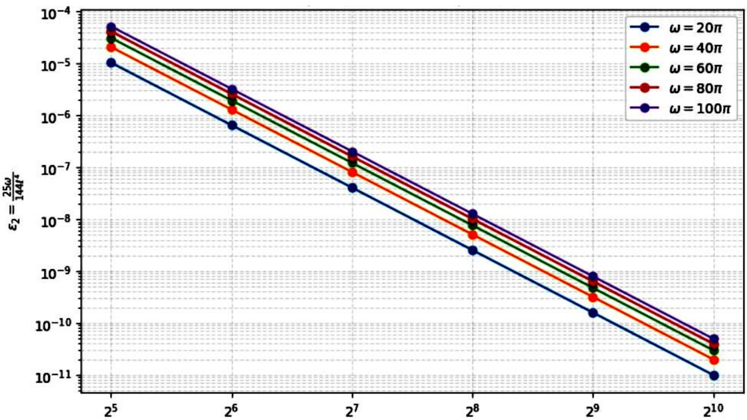

Fig. 2. *Visualization of error values* ε_2

Conclusions. The article provides estimates of approximation errors in numerical integration of rapidly oscillating functions on a class of differentiable functions using sparse grids. Approximation operators were constructed based on O. M. Lytvyn's information operators. A comparative analysis of the accuracy of the cubature formula on different classes of functions showed that the class of differentiability of a function is a determining factor that influences the rate of decrease of the theoretical error of numerical integration. For functions with a higher level of smoothness, a significant increase in accuracy is achieved even when using sparse grids. Thus, the results obtained confirm the need to take into account the smoothness class when choosing the cubature formula and discretization parameters and can serve as a theoretical justification for the use of sparse grids for numerical integration of rapidly oscillating functions of two variables.

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ВПЛИВ ГЛАДКОСТІ НА ТОЧНІСТЬ ЧИСЕЛЬНОГО ІНТЕГРУВАННЯ ШВИДКООСЦИЛЬОВАНИХ ФУНКЦІЙ ДВОХ ЗМІННИХ НА РОЗРІДЖЕНИХ СІТКАХ

Однією з ключових задач у сучасній прикладній математиці, без якої неможливе моделювання та аналіз складних процесів, зокрема в цифровій обробці зображень, є чисельне інтегрування функцій багатьох змінних. Основна проблема чисельного інтегрування функцій багатьох змінних полягає в зростанні обчислювальних витрат зі збільшенням розмірності області інтегрування.

Особливий інтерес становлять методи чисельного інтегрування, розроблені з використанням інформаційних операторів, які відновлюють проміжні значення величин за наявним набором відомих значень функції багатьох змінних в точках, на лініях, площинах, тощо. На основі таких операторів будуються економні схеми інтерполяції функцій декількох змінних. Застосування економних схем в чисельному інтегруванні функцій двох та трьох змінних дозволяє будувати розріджені сітки та обчислювати наближено інтеграли з меншою кількістю даних та із заданою наперед точністю порівняно з класичними методами.

Метою даної статті є демонстрація використання економних схем інтерполяції для наближеного обчислення подвійних інтегралів від швидкоосцильованих функцій загального виду на різних класах гладкості. В роботі проаналізовано вплив порядку диференційовності функції на швидкість спадання теоретичної похибки наближення кубатурної формул. Показано, що зі зростанням гладкості функції покращуються оцінки похибки чисельного інтегрування, що дозволяє ефективно використовувати розріджені сітки без втрати точності. Отримані результати встановлюють кількісний зв'язок між класом диференційовності функції, параметрами дискретизації та частотою осциляцій і можуть бути використані для обґрунтування вибору чисельних методів інтегрування швидкоосцилюючих функцій двох змінних.

Ключові слова: *математичне моделювання процесів, цифрова обробка зображень, чисельне інтегрування, швидкоосцильовані функції багатьох змінних, кубатурна формула, інтерполяція функцій, розріджені сітки.*

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