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NUMERICAL INTEGRATION OF RAPIDLY OSCILLATING FUNCTIONS USING RECONSTRUCTION OPERATORS BASED ON DATA ON LINES

In modern mathematical modelling of physical and technical processes, the problem of processing and analysing functions of many variables, whose values are known on line systems, is relevant. Digital image processing tasks, in particular the numerical integration of oscillating functions, are no exception. In digital image processing tasks, a significant part of the information about the object under study comes in the form of measurements along individual directions or lines, which is characteristic of tomographic methods, remote sensing and visualization systems. The development and application of effective methods for the numerical integration of rapidly oscillating functions based on data on a line system is an important prerequisite for improving the accuracy of reconstruction, filtering, and analysis of digital images.

The research in the article is devoted to the numerical integration of rapidly oscillating functions of several variables. The article presents a cubature formula for the approximate calculation of double integrals of an oscillating exponential function. The cubature formula uses traces on mutually perpendicular lines as function data in its construction. Error estimates are presented for a class of differentiable functions.

Much attention is paid to testing the cubature formula for approximate calculation of double integrals of the osclilating exponential function. The results obtained allow us to explain the choice of parameters and confirm the theoretical error estimates. The numerical experiment is of particular importance because it is the basis for analysing and predicting the behaviour of the method in the three-dimensional case. This is due to the fact that with an increase in dimension, the amount of information required about the integrand increases, the error structure becomes more complex, and the computational costs increase significantly.

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The paper presents a cubature formula for the approximate calculation of triple integrals of an oscillating exponential. The values of the function are given as traces of the function on a system of mutually perpendicular lines. An estimate of the approximation error on a class of differentiable functions is obtained.

Key words: *mathematical modelling of processes, digital image processing, numerical integration of rapidly oscillating functions of several variables, cubature formula, restoration of functions on lines.*

Introduction. In modern mathematical modelling of physical and technical problems, there is an increasing need to process and analyse functions of several variables defined not on the entire domain, but only on limited sets, in particular on lines, sections, and intersections.

In a number of applied situations, such as computer tomography, optical and seismic tomography, digital visualization, aerospace sensing, measurements are only available along certain straight or intersecting lines, due to the physical nature of the experiment or technical limitations of the system. This leads to fundamentally incomplete initial information, which requires special mathematical approaches to restoration, approximation, and numerical integration.

The situation is particularly challenging when the function to be restored or integrated has a rapidly oscillating nature, as is the case, for example, in high-frequency acoustics, quantum mechanics, and electromagnetic modelling. In such cases, traditional methods of numerical integration or interpolation prove to be ineffective or even unsuitable.

The development of new mathematical models and computational methods based on recovery operators from line data opens up the possibility of more accurate modelling of the internal properties of objects, which is critically important in medical diagnostics, materials testing, geophysical exploration, digital image processing, and other fields. Separately, it is also worth noting the issue of constructing mathematical models, where it is necessary to calculate double and triple integrals of rapidly oscillating functions of a general form. This task is more complex, requiring more detailed study and the creation of new approaches to obtain meaningful results.

One effective approach to solving such problems is based on the use of special restoration operators that allow approximating or accurately reconstructing a function over the entire domain based on its values on linear sets. Such operators include O. M. Lytvyn's information operators (interpolation operators) [1, 2]. The role of operators in mathematical modelling of digital signal and image processing, computer and seismic tomography is most fully presented in [3-5].

The problem of approximate calculation of integrals of rapidly oscillating functions of many variables has both a classical solution [6-8] and solutions using various information operators [9, 10]. In [11-14], algorithms for calculating two-dimensional and three-dimensional integrals of fast-oscillating functions of general form are presented. However, the question of approximate calculation of integrals of fast-oscillating functions of three variables of general form in the case of using restoration operators based on data on lines on the class of differentiable functions has not been investigated.

Integration of a rapidly oscillating exponential function on the class $H^{2,1}\left(M,\tilde{M}\right)$**.** Consider the class $H^{2,1}\left(M,\tilde{M}\right)$ – a class of real functions defined on $G=\left[0,1\right]^2$ and such that the first-order partial derivatives with respect to the variables x and y are bounded, i.e.

$$\left|f^{(1,0)}(x,y)\right|\le M,\quad \left|f^{(0,1)}(x,y)\right|\le M,\quad \left|f^{(1,1)}(x,y)\right|\le \tilde{M}\ .$$

For the approximate calculation of the integral of functions of two variables of the form

$$I^2(\omega,f,g)=\int\limits_0^1\int\limits_0^1 f(x,y)\,\mathrm{e}^{i\omega g(x,y)}\,dxdy \tag{1}$$

a cubature formula is proposed using piecewise constant spline interpolation operators based on piecewise constant O. M. Lytvyn's operators. An estimate of the approximation error of the cubature formula is obtained for the class of differentiable functions.

Definition. *By the trace of the function* $u(x,y)$ *on the lines* $x_k=k\Delta-\dfrac{\Delta}{2},\ \ y_j=j\Delta-\dfrac{\Delta}{2},\ \ k,j=\overline{1,\ell},\ \ \Delta=\dfrac{1}{\ell}$ *we mean, respectively, the functions of one variable* $u(x_k,y),\ 0\le y\le 1,\ u(x,y_j),\ 0\le x\le 1.$

Let's use the following symbols:

$$h1_{0k}(x)=\begin{cases}1, & x\in X1_k,\\ 0, & x\notin X1_k,\end{cases}\quad k=\overline{1,\ell_1},\ H1_{0j}(y)=\begin{cases}1, & y\in Y1_j,\\ 0, & y\notin Y1_j,\end{cases}\quad j=\overline{1,\ell_1};$$

$$X1_k=\left[x_{k-1/2},x_{k+1/2}\right],\ Y1_j=\left[y_{j-1/2},y_{j+1/2}\right];$$

$$x_k=k\Delta_1-\Delta_1/2,\ y_j=j\Delta_1-\Delta_1/2,\ k,j=\overline{1,\ell_1},\ \Delta_1=1/\ell_1.$$

$$h2_{0p}(x)=\begin{cases}1, & x\in X2_p,\\ 0, & x\notin X2_p,\end{cases}\quad p=\overline{1,\ell_2},\ H2_{0s}(y)=\begin{cases}1, & y\in Y2_s,\\ 0, & y\notin Y2_s,\end{cases}\quad s=\overline{1,\ell_2};$$

$$X2_p=\left[x_{p-1/2},x_{p+1/2}\right],\ Y2_s=\left[y_{s-1/2},y_{s+1/2}\right];$$

$$x_p=p\Delta_2-\Delta_2/2,\ y_s=s\Delta_2-\Delta_2/2,\ p,s=\overline{1,\ell_2},\ \Delta_2=1/\ell_2.$$

Let us consider the operators

$$Tg(x,y)=\sum_{k=1}^{\ell_1} g(x_k,y)h1_{0k}(x)+\sum_{j=1}^{\ell_1} g(x,y_j)H1_{0j}(y)-$$

$$-\sum_{k=1}^{\ell_1}\sum_{j=1}^{\ell_1} f(x_k,y_j)h1_{0k}(x)H1_{0j}(y).$$

$$\tilde{T}f(x,y)=\sum_{p=1}^{\ell_2} f\left(x_p,y\right)h2_{0p}(x)+\sum_{s=1}^{\ell_2} f\left(x,y_s\right)H2_{0s}(y)-$$

$$-\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2} f\left(x_p,y_s\right)h2_{0p}(x)H2_{0s}(y),$$

If we introduce additional operators

$$T_1g(x,y)=\sum_{k=1}^{\ell_1} g(x_k,y)h1_{0k}(x),\ T_2g(x,y)=\sum_{j=1}^{\ell_1} g(x,y_j)H1_{0j}(y),$$

$$\tilde{T}_1 f(x,y)=\sum_{p=1}^{\ell_2} f\left(x_p,y\right)h2_{0p}(x),\ \tilde{T}_2 f(x,y)=\sum_{s=1}^{\ell_2} f\left(x,y_s\right)H2_{0s}(y)$$

then the following identity holds:

$$Tg=(T_1+T_2+T_1T_2)g\ ,\ \tilde{T}f=(\tilde{T}_1+\tilde{T}_2+\tilde{T}_1\tilde{T}_2)f\ .$$

The cubature formula

$$\Phi^2(\omega,f,g)=\int_0^1\int_0^1 \tilde{T}f(x,y)\,\mathrm{e}^{i\omega Tg(x,y)}\,dxdy \tag{2}$$

is proposed for the approximate calculation of integral (1).

Theorem 1. *[11] Let* $f(x,y),g(x,y)\in H^{2,1}\left(M,\tilde{M}\right)$, *then*

$$\left|I^2(\omega,f,g)-\Phi^2(\omega,f,g)\right|=$$

$$=\left|\int_0^1\int_0^1 f(x,y)\mathrm{e}^{i\omega g(\mathrm{x,y})}\,dxdy-\int_0^1\int_0^1 \tilde{T}f(x,y)\mathrm{e}^{i\omega Tg(\mathrm{x,y})}\,dxdy\right|\le$$

$$\le\frac{\tilde{M}}{16\ell_2{}^2}+\tilde{M}\min\left(2;\frac{\omega\left(\tilde{M}\right)}{16\ell_1{}^2}\right)$$

Consider the integration of a rapidly oscillating exponential function of the form

$$I^2(\omega,g)=\int_0^1\int_0^1 \mathrm{e}^{i\omega g(x,y)}\,dxdy\,. \tag{3}$$

For an approximate calculation of the integral (3), the following quadrature formula is proposed

$$\Phi^2(\omega, g) = \int_0^1 \int_0^1 \mathrm{e}^{i\omega Tg(x,y)}\, dxdy\,. \tag{4}$$

Theorem 2. *Let* $g(x,y) \in H^{2,1}\left(M, \tilde{M}\right)$, *then*

$$\left| I^2(\omega, g) - \Phi^2(\omega, g) \right| =$$

$$= \left| \int_0^1 \int_0^1 e^{i\omega g(x,y)} dxdy - \int_0^1 \int_0^1 e^{i\omega Tg(x,y)} dxdy \right| \le \min\left(2; \frac{\omega\left(\tilde{M}\right)}{16\ell_1^{\,2}} \right).$$

Integration of a rapidly oscillating exponential function on the class $H^{3,1}\left(M, \bar{M}, \tilde{M}\right)$. Consider $H_1^{3,1}\left(M, \overline{M}, \tilde{M}\right)$ – a class of real functions defined on $G = [0,1]^3$ and such that

$$\left| g^{(1,0,0)}(x,y,z) \right| \le M, \quad \left| g^{(0,1,0)}(x,y,z) \right| \le M, \quad \left| g^{(0,0,1)}(x,y,z) \right| \le M\,,$$

$$\left| g^{(1,1,0)}(x,y,z) \right| \le \overline{M}, \quad \left| g^{(1,0,1)}(x,y,z) \right| \le \overline{M}, \quad \left| g^{(0,1,1)}(x,y,z) \right| \le \overline{M}\,,$$

$$\left| g^{(1,1,1)}(x,y,z) \right| \le \tilde{M}\,.$$

Definition 2. *By the trace of the function* $v(x,y,z)$ *on the lines*

$$\Big\{ (x,y,z):\ x = x_k,\ y = y_j, \quad x_k = k\Delta - \frac{\Delta}{2},$$

$$y_j = j\Delta - \frac{\Delta}{2}, \Delta = \frac{1}{\ell},\ k, j = \overline{1,\ell}, 0 \le z \le 1 \Big\}$$

we mean $v(x_k, y_j, z),\quad 0 \le z \le 1$. *The traces of the function on other lines are defined analogously.*

Let us introduce the following notation

$$X_k = \left[x_{k-1/2}, x_{k+1/2} \right],\ Y_j = \left[y_{j-1/2}, y_{j+1/2} \right], \quad Z_s = \left[z_{s-1/2}, z_{s+1/2} \right],$$

$$\tilde{X}_{\tilde{k}} = \left[\tilde{x}_{\tilde{k}-1/2}, \tilde{x}_{\tilde{k}+1/2} \right],\ \tilde{Y}_{\tilde{j}} = \left[\tilde{y}_{\tilde{j}-1/2}, \tilde{y}_{\tilde{j}+1/2} \right], \quad \tilde{Z}_{\tilde{s}} = \left[\tilde{z}_{\tilde{s}-1/2}, \tilde{z}_{\tilde{s}+1/2} \right],$$

$$h_{1k}^0(x) = \begin{cases} 1, x \in X_k, \\ 0, x \notin X_k, \end{cases} \quad h_{2j}^0(y) = \begin{cases} 1, y \in Y_j, \\ 0, y \notin Y_j, \end{cases} \quad h_{3s}^0(z) = \begin{cases} 1, z \in Z_s, \\ 0, z \notin Z_s, \end{cases}$$

$$\tilde{h}_{1\tilde{k}}^0(x) = \begin{cases} 1, x \in \tilde{X}_{\tilde{k}}, \\ 0, x \notin \tilde{X}_{\tilde{k}}, \end{cases} \quad \tilde{h}_{2\tilde{j}}^0(y) = \begin{cases} 1, y \in \tilde{Y}_{\tilde{j}}, \\ 0, y \notin \tilde{Y}_{\tilde{j}}, \end{cases} \quad \tilde{h}_{3\tilde{s}}^0(z) = \begin{cases} 1, z \in \tilde{Z}_{\tilde{s}}, \\ 0, z \notin \tilde{Z}_{\tilde{s}}, \end{cases},$$

$$x_k = k\Delta - \frac{\Delta}{2},\ y_j = j\Delta - \frac{\Delta}{2}, \quad z_s = s\Delta - \frac{\Delta}{2}, \quad \Delta = \frac{1}{\ell},\ k, j, s = \overline{1,\ell}\,,$$

$$\tilde{x}_{\tilde{k}} = \tilde{k}\Delta_1 - \frac{\Delta_1}{2},\ \tilde{y}_{\tilde{j}} = \tilde{j}\Delta_1 - \frac{\Delta_1}{2},\ \tilde{z}_{\tilde{s}} = \tilde{s}\Delta_1 - \frac{\Delta_1}{2}, \quad \Delta_1 = \frac{1}{\ell^{3/2}}, \quad \tilde{k}, \tilde{j}, \tilde{s} = \overline{1, \ell^{3/2}} .$$

Let us consider the operators

$$J_1 f(x,y,z) = \sum_{k=1}^{\ell} f(x_k, y, z) h_{1k}^0(x)\ ,\ J_2 f(x,y,z) = \sum_{j=1}^{\ell} f(x, y_j, z) h_{2j}^0(y)\ ,$$

$$J_3 f(x,y,z) = \sum_{s=1}^{\ell} f(x, y, z_s) h_{3s}(z)\ ,$$

$$\tilde{J}_1 f(x,y,z) = \sum_{\tilde{k}=1}^{\ell^{3/2}} f(\tilde{x}_{\tilde{k}}, y, z) \tilde{h}_{1\tilde{k}}^0(x)\ ,\ \tilde{J}_2 f(x,y,z) = \sum_{\tilde{j}=1}^{\ell^{3/2}} f(x, \tilde{y}_{\tilde{j}}, z) \tilde{h}_{2\tilde{j}}^0(y),$$

$$\tilde{J}_3 f(x,y,z) = \sum_{\tilde{s}=1}^{\ell^{3/2}} f(x, y, \tilde{z}_{\tilde{s}}) \tilde{h}_{3\tilde{s}}^0(z)\ .$$

Let us consider the piecewise constant interflatation (O. M. Lytvyn's) operator

$$Ef(x,y,z) = J_1 f\left(x,y,z\right) + J_2 f\left(x,y,z\right) + J_3 f\left(x,y,z\right) -$$
$$- J_1 J_2 f\left(x,y,z\right) - J_2 J_3 f\left(x,y,z\right) - J_1 J_3 f\left(x,y,z\right) + J_1 J_2 J_3 f\left(x,y,z\right)$$

for which the properties

$$Ef\left(x_k, y, z\right) = f\left(x_k, y, z\right), \quad k = \overline{1,\ell}, \quad Ef\left(x, y_j, z\right) = f\left(x, y_j, z\right), \quad j = \overline{1,\ell},$$

$$Ef\left(x, y, z_S\right) = f\left(x, y, z_S\right), \quad s = \overline{1,\ell}\ .$$

Consider the piecewise constant interlining operator constructed on the basis of interfletation

$$\tilde{E}f\left(x,y,z\right) = J_1\tilde{J}_2 f\left(x,y,z\right) + J_1\tilde{J}_3 f\left(x,y,z\right) - J_1\tilde{J}_2\tilde{J}_3 f\left(x,y,z\right) + J_2\tilde{J}_1 f\left(x,y,z\right) +$$
$$+ J_2\tilde{J}_3 f\left(x,y,z\right) - J_2\tilde{J}_1\tilde{J}_3 f\left(x,y,z\right) + J_3\tilde{J}_1 f\left(x,y,z\right) + J_3\tilde{J}_2 f\left(x,y,z\right) - J_3\tilde{J}_1\tilde{J}_2 f\left(x,y,z\right) -$$
$$- J_1 J_2 f\left(x,y,z\right) - J_1 J_3 f\left(x,y,z\right) - J_2 J_3 f\left(x,y,z\right) + J_1 J_2 J_3 f\left(x,y,z\right).$$

To calculate the integral

$$I(g,\omega) = \int_0^1 \int_0^1 \int_0^1 \mathrm{e}^{i\omega g(x,y,z)}\, dxdydz,\ \left|\omega\right| \ge 2\pi, \tag{5}$$

the following formula is proposed:

$$\Phi(g,\omega) = \int_0^1 \int_0^1 \int_0^1 e^{i\omega \tilde{E} g(x,y,z)} dxdydz\ . \tag{6}$$

For an approximate calculation of the integral (5) using formula (6), we obtain an approximation error on the class of differentiable functions. For the proof, we will use the estimate obtained in Theorem 3.

Theorem 3. *[15] When calculating*

$$I(g)=\int_0^1\int_0^1\int_0^1 g(x,y,z)dxdydz$$

using the cubature formula

$$\Phi\left(g\right)=\int_0^1\int_0^1\int_0^1 Eg(x,y,z)dxdydz$$

the estimate $\left|I(g)-\Phi(g)\right|\le\dfrac{\tilde{M}}{64\ell_1^3}$ *will be satisfied.*

Theorem 4. *For the cubature formula*

$$\Phi(g,\omega)=\int_0^1\int_0^1\int_0^1 e^{i\omega\tilde{E}g(x,y,z)}dxdydz$$

the following approximation error estimate holds

$$\left|I(g,\omega)-\varPhi(g,\omega)\right|\le\min\left(2,\frac{\omega\tilde{M}}{64\ell_1^3}+3M\cdot\omega\cdot\frac{\Delta_1{}^3}{16}\right).$$

Proof. From Theorem 3, we use the estimate

$$\int_0^1\int_0^1\int_0^1\left|g(x,y,z)-Eg(x,y,z)\right|dxdydz\ \le\frac{\tilde{M}}{64\ell_1{}^3}.$$

We find the estimate $\int_0^1\int_0^1\int_0^1\left|Eg(x,y,z)-\tilde{E}g(x,y,z)\right|dxdydz$:

$$\int_0^1\int_0^1\int_0^1\left|Eg(x,y,z)-\tilde{E}g(x,y,z)\right|dxdydz\ \le$$

$$\le\int_0^1\int_0^1\int_0^1\Big|E_1g\left(x,y,z\right)+E_2g\left(x,y,z\right)+E_3g\left(x,y,z\right)-E_1E_2g\left(x,y,z\right)-$$
$$-E_2E_3g\left(x,y,z\right)-E_1E_3g\left(x,y,z\right)+E_1E_2E_3g\left(x,y,z\right)-E_1\tilde{E}_2g\left(x,y,z\right)-$$
$$-E_1\tilde{E}_3g\left(x,y,z\right)+E_1\tilde{E}_2\tilde{E}_3g\left(x,y,z\right)-E_2\tilde{E}_1g\left(x,y,z\right)-E_2\tilde{E}_3f\left(x,y,z\right)+$$
$$+E_2\tilde{E}_1\tilde{E}_3f\left(x,y,z\right)-\tilde{E}_3\tilde{E}_1f\left(x,y,z\right)-E_3\tilde{E}_2f\left(x,y,z\right)+E_3\tilde{E}_1\tilde{E}_2f\left(x,y,z\right)+$$
$$+E_1E_2f\left(x,y,z\right)+E_1E_3f\left(x,y,z\right)+E_2E_3f\left(x,y,z\right)-E_1E_2E_3f\left(x,y,z\right)\Big|dxdydz=$$

$$\le\int_0^1\int_0^1\int_0^1\Big|E_1g\left(x,y,z\right)+E_2g\left(x,y,z\right)+E_3g\left(x,y,z\right)-E_1\tilde{E}_2g\left(x,y,z\right)-E_1\tilde{E}_3g\left(x,y,z\right)+$$

$$+E_1\tilde{E}_2\tilde{E}_3 g(x,y,z) - E_2\tilde{E}_1 g(x,y,z) - E_2\tilde{E}_3 g(x,y,z) + E_2\tilde{E}_1\tilde{E}_3 g(x,y,z) -$$

$$-E_3\tilde{E}_1 g(x,y,z) - E_3\tilde{E}_2 g(x,y,z) + E_3\tilde{E}_1\tilde{E}_2 g(x,y,z)\Big|\, dxdydz =$$

$$= \int_0^1 \int_0^1 \int_0^1 \Big| \left(E_1 - E_1\tilde{E}_2 - E_1\tilde{E}_3 + E_1\tilde{E}_2\tilde{E}_3\right) g(x,y,z) +$$

$$+\left(E_2 - E_2\tilde{E}_1 - E_2\tilde{E}_3 + E_2\tilde{E}_1\tilde{E}_3\right) g(x,y,z) +$$

$$+\left(E_3 - E_3\tilde{E}_1 - E_3\tilde{E}_2 + E_3\tilde{E}_1\tilde{E}_2\right) g(x,y,z)\Big|\, dxdydz \le$$

$$\le \int_0^1 \int_0^1 \int_0^1 \left| \left(E_1 - E_1\tilde{E}_2 - E_1\tilde{E}_3 + E_1\tilde{E}_2\tilde{E}_3\right) g(x,y,z)\right| dxdydz +$$

$$+\int_0^1 \int_0^1 \int_0^1 \left| \left(E_2 - E_2\tilde{E}_1 - E_2\tilde{E}_3 + E_2\tilde{E}_1\tilde{E}_3\right) g(x,y,z)\right| dxdydz +$$

$$+\int_0^1 \int_0^1 \int_0^1 \left| \left(E_3 - E_3\tilde{E}_1 - E_3\tilde{E}_2 + E_3\tilde{E}_1\tilde{E}_2\right) g(x,y,z)\right| dxdydz \le$$

$$\le \sum_{k=1}^{\ell} \sum_{\tilde{j}=1}^{\ell^{3/2}} \sum_{\tilde{s}=1}^{\ell^{3/2}} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}} \int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}} \Big| g(x_k, y, z) - g\left(x_k, \tilde{y}_{\tilde{j}}, z\right) -$$

$$-g\left(x_k, y, \tilde{z}_{\tilde{s}}\right) + g\left(x_k, \tilde{y}_{\tilde{j}}, \tilde{z}_{\tilde{s}}\right)\Big|\, dxdydz +$$

$$+\sum_{j=1}^{\ell} \sum_{\tilde{k}=1}^{\ell^{3/2}} \sum_{\tilde{s}=1}^{\ell^{3/2}} \int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}} \left| g\left(x, y_j, z\right) - g\left(\tilde{x}_{\tilde{k}}, y_j, z\right) - g\left(x, y_j, \tilde{z}_{\tilde{s}}\right) + g\left(\tilde{x}_{\tilde{k}}, y_j, \tilde{z}_{\tilde{s}}\right)\right| dxdydz +$$

$$+\sum_{s=1}^{\ell} \sum_{\tilde{k}=1}^{\ell^{3/2}} \sum_{\tilde{j}=1}^{\ell^{3/2}} \int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}} \int_{\tilde{y}_{\tilde{j}-}\frac{1}{2}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}} \int_{z_{s-\frac{1}{2}}}^{z_{s+\frac{1}{2}}} \left| g\left(\tilde{x}_{\tilde{k}}, y, z_s\right) - g\left(x, \tilde{y}_{\tilde{j}}, z_s\right) - g\left(x, y, z_s\right) + g\left(\tilde{x}_{\tilde{k}}, \tilde{y}_{\tilde{j}}, z_s\right)\right| dxdydz \le$$

$$\le \sum_{k=1}^{\ell} \sum_{\tilde{j}=1}^{\ell^{3/2}} \sum_{\tilde{s}=1}^{\ell^{3/2}} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}} \int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}} \left| \int_{\tilde{y}_{\tilde{j}}}^{y} \int_{\tilde{z}_{\tilde{s}}}^{z} g^{(0,1,1)}(x_k, \eta, \zeta)\, d\eta\, d\zeta \right| dxdydz +$$

$$+\sum_{j=1}^{\ell} \sum_{\tilde{k}=1}^{\ell^{3/2}} \sum_{\tilde{s}=1}^{\ell^{3/2}} \int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}} \left| \int_{\tilde{x}_{\tilde{k}}}^{x} \int_{\tilde{z}_{\tilde{s}}}^{z} g^{(1,0,1)}(\xi, y_j, \zeta)\, d\xi\, d\zeta \right| dxdydz +$$

$$+\sum_{s=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\tilde{j}=1}^{\ell^{3/2}}\int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}}\int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}}\int_{z_{s-\frac{1}{2}}}^{z_{s+\frac{1}{2}}}\left|\int_{\tilde{x}_{\tilde{k}}}^{x}\int_{\tilde{y}_{\tilde{j}}}^{y} g^{(1,1,0)}(\xi,\eta,z_s)d\xi d\eta\right|dxdydz \le$$

$$\le \bar{M}\sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^{3/2}}\sum_{\tilde{s}=1}^{\ell^{3/2}}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}}\int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}}\left|\int_{\tilde{y}_{\tilde{j}}}^{y}\int_{\tilde{z}_{\tilde{s}}}^{z} d\eta d\zeta\right|dxdydz +$$

$$+\bar{M}\sum_{j=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\tilde{s}=1}^{\ell^{3/2}}\int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}}\left|\int_{\tilde{x}_{\tilde{k}}}^{x}\int_{\tilde{z}_{\tilde{s}}}^{z} d\xi d\zeta\right|dxdydz +$$

$$+\bar{M}\sum_{s=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\tilde{j}=1}^{\ell^{3/2}}\int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}}\int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}}\int_{z_{s-\frac{1}{2}}}^{z_{s+\frac{1}{2}}}\left|\int_{\tilde{x}_{\tilde{k}}}^{x}\int_{\tilde{y}_{\tilde{j}}}^{y} d\xi d\eta\right|dxdydz =$$

$$\le \bar{M}\sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^{3/2}}\sum_{\tilde{s}=1}^{\ell^{3/2}}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}}\int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}}\left|y-\tilde{y}_{\tilde{j}}\right|\left|z-\tilde{z}_{\tilde{s}}\right|dxdydz +$$

$$+\bar{M}\sum_{j=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\tilde{s}=1}^{\ell^{3/2}}\int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{\tilde{z}_{\tilde{s}-\frac{1}{2}}}^{\tilde{z}_{\tilde{s}+\frac{1}{2}}}\left|x-\tilde{x}_{\tilde{k}}\right|\left|z-\tilde{z}_{\tilde{s}}\right|dxdydz +$$

$$+\bar{M}\sum_{s=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^{3/2}}\sum_{\tilde{j}=1}^{\ell^{3/2}}\int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}}\int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}}\int_{z_{s-\frac{1}{2}}}^{z_{s+\frac{1}{2}}}\left|x-\tilde{x}_{\tilde{k}}\right|\left|y-\tilde{y}_{\tilde{j}}\right|dxdydz =$$

$$=\bar{M}\sum_{k=1}^{\ell_1}\sum_{\tilde{j}=1}^{\ell_1^{3/2}}\sum_{\tilde{s}=1}^{\ell_1^{3/2}}\Delta_1\cdot\frac{\tilde{\Delta}_1^2}{4}\cdot\frac{\tilde{\Delta}_1^2}{4}+\bar{M}\sum_{k=1}^{\ell_1}\sum_{\tilde{j}=1}^{\ell_1^{3/2}}\sum_{\tilde{s}=1}^{\ell_1^{3/2}}\Delta_1\cdot\frac{\tilde{\Delta}_1^2}{4}\cdot\frac{\tilde{\Delta}_1^2}{4}+\bar{M}\sum_{k=1}^{\ell_1}\sum_{\tilde{j}=1}^{\ell_1^{3/2}}\sum_{\tilde{s}=1}^{\ell_1^{3/2}}\Delta_1\cdot\frac{\tilde{\Delta}_1^2}{4}\cdot\frac{\tilde{\Delta}_1^2}{4}=$$

$$=3\bar{M}\cdot\ell_1\cdot\ell_1^{\frac{3}{2}}\cdot\ell_1^{\frac{3}{2}}\cdot\Delta_1\cdot\frac{\tilde{\Delta}_1^2}{4}\cdot\frac{\tilde{\Delta}_1^2}{4}=3\bar{M}\cdot\ell_1^4\cdot\Delta_1\cdot\frac{\tilde{\Delta}_1^4}{16}=3\bar{M}\cdot\frac{\Delta_1^3}{16}.$$

We use the following fact:

$$e^{i\omega g(x,y,z)}-e^{i\omega\tilde{E}g(x,y,z)}=$$

$$=2\sin\frac{\omega g(x,y,z)-\omega\tilde{E}g(x,y,z)}{2}e^{i\frac{\omega}{2}\left(g(x,y,z)+\tilde{E}g(x,y,z)\right)}.$$

Let us estimate the approximation error

$$\left|I(g,\omega)-\Phi(g,\omega)\right| \le \left|\int_0^1\int_0^1\int_0^1 e^{i\omega g(x,y,z)}dxdydz - \int_0^1\int_0^1\int_0^1 e^{i\omega \tilde{E}g(x,y,z)}dxdydz\right| \le$$

$$\le \int_0^1\int_0^1\int_0^1 \left|e^{i\omega g(x,y,z)} - e^{i\omega \tilde{E}g(x,y,z)}\right| dxdydz =$$

$$= \int_0^1\int_0^1\int_0^1 \left|2\sin\frac{\omega g(x,y,z)-\omega\tilde{E}g(x,y,z)}{2}\, e^{i\frac{\omega}{2}\left(g(x,y,z)+\tilde{E}g(x,y,z)\right)}\right| dxdydz \le$$

$$\le 2\int_0^1\int_0^1\int_0^1 \left|\sin\frac{\omega(g(x,y,z)-\tilde{E}g(x,y,z))}{2}\right| dxdydz \le$$

$$\le 2\int_0^1\int_0^1\int_0^1 \min\left(1,\frac{\omega\left|g(x,y,z)-\tilde{E}g(x,y,z)\right|}{2}\right) dxdydz \le$$

$$\le 2\min\left(\int_0^1\int_0^1\int_0^1 dxdydz, \int_0^1\int_0^1\int_0^1 \frac{\omega\left|g(x,y,z)-\tilde{E}g(x,y,z)\right|}{2} dxdydz\right) \le$$

$$\le 2\min\left(\int_0^1\int_0^1\int_0^1 dxdydz, \frac{\omega}{2}\int_0^1\int_0^1\int_0^1 \left|g(x,y,z)-Eg(x,y,z)+\right.\right.$$

$$\left.\left.+Eg(x,y,z)-\tilde{E}g(x,y,z)\right| dxdydz\right) \le$$

$$\le 2\min\left(\int_0^1\int_0^1\int_0^1 dxdydz\,, \frac{\omega}{2}\left(\int_0^1\int_0^1\int_0^1 \left|g(x,y,z)-Eg(x,y,z)\right| dxdydz\right.\right. +$$

$$\left.\left.+\int_0^1\int_0^1\int_0^1 \left|Eg(x,y,z)-\tilde{E}g(x,y,z)\right| dxdydz\right)\right) \le$$

$$\le 2\min\left(1,\frac{\omega}{2}\left(\frac{\tilde{M}}{64\ell_1^3}+3\bar{M}\cdot\frac{\Delta_1^3}{16}\right)\right) = \min\left(2,\frac{\omega\tilde{M}}{64\ell_1^3}+3\bar{M}\cdot\omega\cdot\frac{\Delta_1^3}{16}\right).$$

The theorem is proven.

Results of the computational experiment. When studying numerical methods for three-dimensional oscillatory integrals, the properties that determine the complexity of the problem and the nature of the error are already evident in the two-dimensional case. In this regard, it is advisable to conduct a numerical experiment for a function of two variables, and the results obtained can be used for a qualitative assessment of the behaviour of the method in the three-dimensional case.

Let us consider the function $g(x,y,z)=e^{(xyz)^{\frac{3}{2}}}$

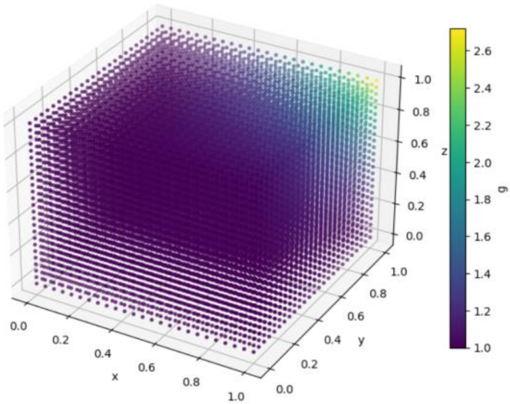

Fig. 1. *Three-dimensional point visualization of the function values $g(x,y,z)$ with colour coding*

The function $g(x,y,z)=e^{(xyz)^{\frac{3}{2}}}$ is a generalization of the two-dimensional function $g(x,y)=e^{(xy)^{\frac{3}{2}}}$.

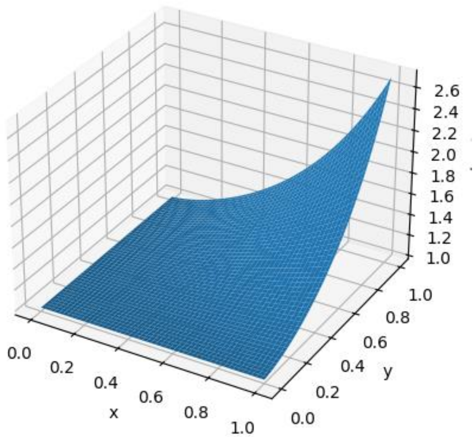

Fig. 2. *3D graph of the function $g(x,y)$*

A comparison of the graph of the function of two variables (Fig. 2) and the corresponding cross-sections of the function of three variables when the value of one of the variables is fixed shows their complete correspondence (Figs. 3-5).

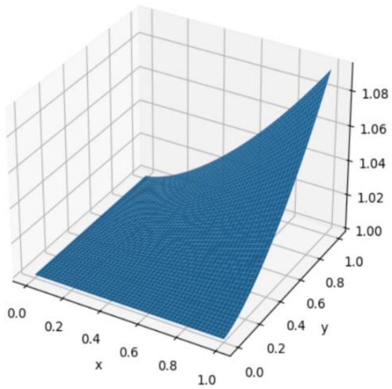

Fig. 3. *Function* $g(x, y, z)$ *when* $z_0 = 0.5$

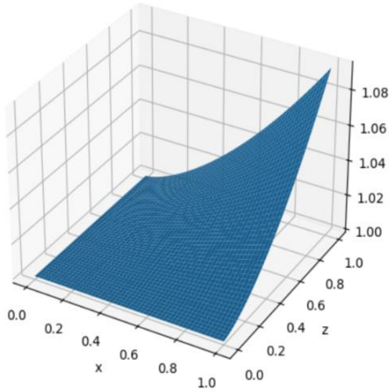

Fig. 4. *Function* $g(x, y, z)$ *when* $y_0 = 0.5$

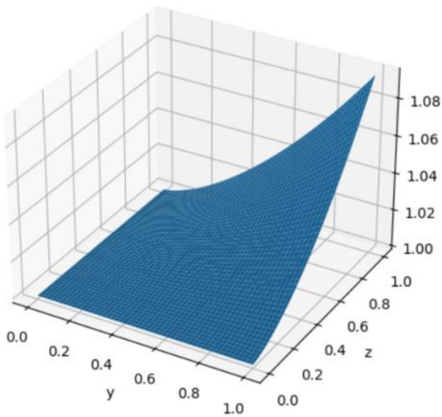

Fig. 5. *Function* $g(x, y, z)$ *at* $x_0 = 0.5$

For the function of two variables, a numerical experiment was conducted to investigate in detail the influence of the parameters ω, $\ell_1 = \ell_2 = \ell$, and $\tilde{M}$ on the accuracy of numerical integration.

The experiment was conducted for the function $g(x, y) = e^{(xy)^{\frac{3}{2}}}$. The function $g(x, y)$ belongs to the class $H^{2,1}\left(M, \tilde{M}\right)$, where $M = \frac{3}{2}e$, $\tilde{M} = \frac{9}{2}e$.

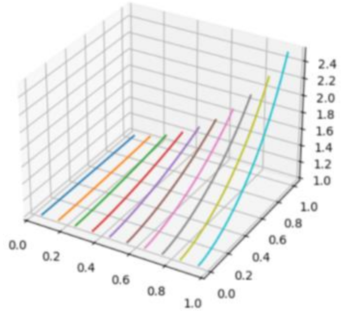

Fig. 6. *Traces of the function on the lines* $x = x_k, k = 1,\ldots,\ell$

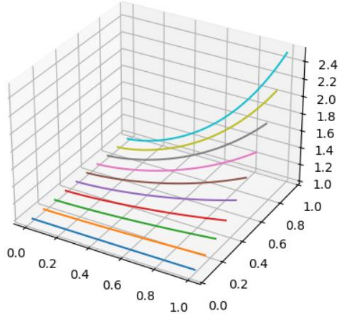

Fig. 7. *Traces of the function on the lines* $y = y_j, j = 1,\ldots,\ell$

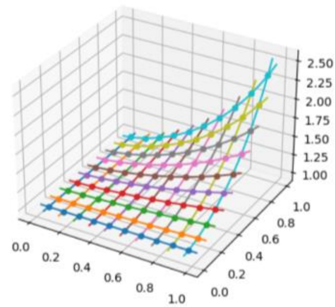

Fig. 8. *Function values at nodal points* $x = x_k, y = y_j, k, j = 1,\ldots,\ell$

Figs. 9-10 show three-dimensional graphs of the real and imaginary parts of the subintegral function $e^{i\omega g(x,y)}$ at $\omega = 10\pi$. From these figures, we can see that even with a smooth phase function $g(x,y)$, the subintegral function acquires a pronounced oscillatory character.

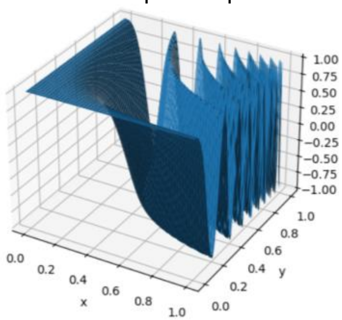

Fig. 9. *3D graph of* $\mathrm{Re}\left(e^{i\omega g(x,y)}\right)$, $\omega = 10\pi$

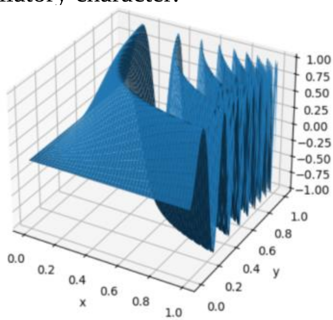

Fig. 10. *3D graph of* $\mathrm{Im}\left(e^{i\omega g(x,y)}\right)$, $\omega = 10\pi$

As the parameter ω increases, the frequency of oscillations increases. We can see this in Figs. 11-12 at $\omega = 160\pi$.

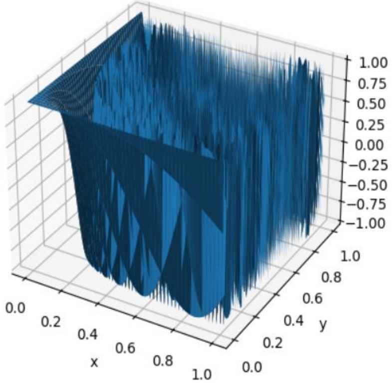

Fig. 11. *3D graph* $\mathrm{Re}\left(e^{i\omega g(x,y)}\right)$, $\omega = 160\pi$

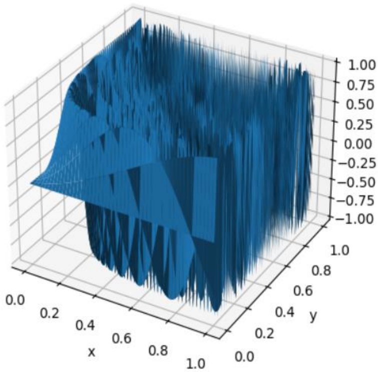

Fig. 12. *3D graph* $\mathrm{Im}\left(e^{i\omega g(x,y)}\right)$, $\omega = 160\pi$

For a detailed analysis of the influence of the oscillation parameter ω and the division ℓ on the accuracy of numerical integration and in order to compare the obtained error estimates with theoretical ones, a series of numerical experiments was performed. The tables show the results of calculations for different values of ω .

Table 1

Calculation results for $\omega = 10\pi$ *, reference value of the integral* $I^2(\omega, g) = 0.2278189646082499 + 0.1991111463311658i$

ℓ	$\Phi^2(\omega, g)$	$\left\|I^2(\omega,g) - \Phi^2(\omega,g)\right\|$	ε_t theoretical error
32	0.2278166992662413 + 0.1991155315055722i	4.9357399638e-06	2.345508e-02
64	0.2278188638642554 + 0.1991112858753887i	1.7211026284e-07	5.863770e-03
12	0.2278189588368593 + 0.1991111539807172i	9.5825146008e-09	1.465943e-03
256	0.2278189642550256 + 0.1991114679943563i	5.8250566720e-10	3.66486e-04
512	0.2278189645862877 + 0.1991114463598729i	3.6144612238e-11	9.162140e-05
1024	0.2278189646068789 + 0.1991114463329548i	2.2539628907e-12	2.290540e-05

Table 2

Calculation results at $\omega = 40\pi$ *, reference value of the integral*

$$I^2(\omega, g) = 0.1067062929189802 + 0.1085966978209356i$$

ℓ	$\Phi^2(\omega, g)$	$\left\|I^2(\omega, g) - \Phi^2(\omega, g)\right\|$	ε_t theoretical error
32	0.1066203629718906 + 0.1086594747825144i	1.0641852617e-04	9.382032e-02
64	0.1067460877089656 + 0.1085472446431358i	6.3476311365e-05	2.345508e-02
12	0.1067063563388021 + 0.1085967474983053i	8.0560007814e-08	5.863770e-03
256	0.1067062950844278 + 0.1085966995478552i	2.7697318077e-09	1.465943e-03
512	0.1067062930401892 + 0.1085966979169320i	1.5461865937e-10	3.664856e-04
1024	0.1067062929263625 + 0.1085966978267644i	9.4059812274e-12	9.162140e-05

Table 3

Calculation results at $\omega = 80\pi$ *, reference value of the integral*

$$I^2(\omega, g) = 0.0723874992954767 + 0.0775883379028776i$$

ℓ	$\Phi^2(\omega, g)$	$\left\|I^2(\omega, g) - \Phi^2(\omega, g)\right\|$	ε_t theoretical error
32	0.0723614746497498 + 0.0777878595754382i	2.0121177900e-04	1.876406e-01
64	0.0723318856918027 + 0.0775744279864674i	5.7326771130e-05	4.691016e-02
12	0.0723790740671419 + 0.0776040305528926i	1.7811337316e-05	1.172754e-02
256	0.0723874891873205 + 0.0775883389270071i	1.0159904640e-08	2.931885e-03
512	0.0723874989439382 + 0.0775883379230748i	3.5211826050e-10	7.329713e-04
1024	0.0723874992757894 + 0.0775883379038215i	1.9709895301e-11	1.832428e-04

Table 4

Calculation results at $\omega = 160\pi$ *, reference value of the integral*

$$I^2(\omega, g) = 0.0488723915931327 + 0.0546273706291931i$$

ℓ	$\Phi^2(\omega, g)$	$\left\|I^2(\omega, g) - \Phi^2(\omega, g)\right\|$	ε_t theoretical error
32	0.0492121569121379 + 0.0545134278246065i	3.5836215581e-04	3.752813e-01
64	0.0488741262706387 + 0.0546801209405205i	5.2778825784e-05	9.382032e-02

Continuation of the table 4

12	0.0488798958008194 + 0.0546252931893292i	7.7864555090e-06	2.345508e-02
256	0.0488723919111890 + 0.0546273718372643i	2.4391440926e-06	5.863770e-03
512	0.0488723919111890 + 0.0546273718372643i	1.2492380771e-09	1.465943e-03
1024	0.0488723916041078 + 0.0546273706703357i	4.2581327138e-11	3.664856e-04

Conclusions. The article is devoted to the numerical integration of rapidly oscillating functions of several variables, which is an important direction in mathematical modelling of processes, in particular in digital image processing problems. Cubic formulas for the approximate calculation of double and triple integrals of oscillating exponentials are presented. Cubic formulas use traces on mutually perpendicular lines as function data in their construction. Error estimates are presented for a class of differentiable functions. A numerical experiment presents an analysis of the method for a function of two variables, which makes it possible to estimate the potential capability of the algorithm in a three-dimensional case.

References:

1. Lytvyn O. N., Sergienko I. V. New Information Operators in Mathematical Modeling (A Review). *Cybernetics and Systems Analysis*. 2018. № 54 (1). P. 21-30. DOI: 10.1007/s10559-018-0004-5.
2. Sergienko I. V., Zadiraka V. K., Lytvyn O. M. Elements of the General Theory of Optimal Algorithms. *Springer*. 2021. P. 378. DOI: 10.1007/978-3-030-90908-6
3. Lytvyn O. N., Pershina Y. I., Sergienko I. V. Estimation of discontinuous functions of two variables with unknown discontinuity lines (rectangular elements). *Cybernet. Syst. Anal*. 2014. № 5 (4). P. 594-602. DOI:10.1007/s10559-014-9647-z.
4. Pershyna I. Mathematical Modeling of 2D Discontinuous Objects by New Information Operators. *International Conference on Reliable Systems Engineering (ICoRSE). Lecture Notes in Networks and Systems*. 2023. Vol. 762. DOI:10.1007/978-3-031-40628-7_33.
5. Nechuiviter O. P., Iarmosh O. V., Kovalchuk K. H. Numerical calculation of multidimensional integrals depended on input information about the function in mathematical modelling of technical and economic processes. *IOP Conference Series: Materials Science and Engineering*. Vol. 1031 (1). Vol. 012059. DOI: 10.1088/1757-899X/1031/1/012059.
6. Zadiraka V. K., Melnikova S. S., Luts L. V. Optimal quadrature evaluation of integrals of rapidly oscillating functions in Lipschitz interpolation class in case of strong oscillation. *Cybern Syst Anal*. 2010. Vol. 46. P. 264-270. DOI: 10.1007/s10559-010-9203-4
7. Zadiraka V. K., Melnikova S. S., Luts L. V. Optimal integration of rapidly oscillating functions in the class W2, L, N with the use of different information

operators. *Cybern. Syst. Analysis.* 2013. Vol. 49. № 2. P. 229-238. DOI: 10.1007/s10559-013-9504-5.
8. Luts L. V. Optimal Calculation of Integrals of Rapidly Oscillating Functions for Some Classes of Differential Functions. *Cybernetics and Systems Analysis.* 2024. № 60. P. 276-284. DOI: 10.1007/s10559-024-00668-5.
9. Lytvyn O. N., Nechuyviter O. P. 3D-Fourier Coefficients on the Class of Differentiable Functions and Spline Interflatation. *Journal of Automation and Information Sciences*. 2012. Vol. 44. Is. 3. P. 45-56. DOI: 10.1615/JAutomatInfScien.v44.i3.40.
10. Lytvyn O. M., Nechuiviter O. P. Approximate Calculation of Triple Integrals of Rapidly Oscillating Functions with the Use of Lagrange Polynomial Interflation. *Cybernetics and Systems Analysis*. 2014. № 50 (3). P. 410-418. DOI: 10.1007/s10559-014-9629-1.
11. Lytvyn O. M., Nechuiviter O., Pershyna Y., Mezhuyev V. Input Information in the Approximate Calculation of Two-Dimensional Integral from Highly Oscillating Functions (Irregular Case). *Recent Developments in Data Science and Intelligent Analysis of Information, Proceedings of the XVIII International Conference on Data Science and Intelligent Analysis of Information.* Kyiv, 2018. P. 365-373. DOI: 10.1007/ 978-3-319-97885-7_36.
12. Nechuiviter O. P. Cubature formula for approximate calculation integral of highly oscillating function of tree variables (irregular case). *Radio Electronics, Computer Science, Control.* 2020. Vol. 4. P. 65-73. DOI: 10.15588/1607-3274-2020-4-7.
13. Nechuiviter O. P. Approximate Calculation of Double Integrals of Highly Oscillating Functions of General Type Using the Cubature Formula of Optimal Order of Accuracy on a Class of Differentiable Functions. *Cybern. Syst. Anal.* 2025. Vol. 61. P. 589-595. DOI: 10.1007/s10559-025-00794-8.
14. Khurdei Y., Ivanov V. Cubature Formulas for Oscillatory Integrals with Given Function Traces on Lines. *Cybernetics and Computer Technologies.* № 3. P. 59-67. DOI: 10.34229/2707-451X.25.3.5.
15. Nechuiviter O. P. Ivanov S. S., Kovalchuk K. G. New information operators in problems of numerical integration of functions of three variables. *Bulletin of the National Technical University «KhPI». Series: Mathematical modeling in engineering and technologies.* 2022. № 1. P. 82-91. DOI: 10.20998/2222-0631.2022.01.10.

ЧИСЕЛЬНЕ ІНТЕГРУВАННЯ ШВИДКООСЦИЛЬОВАНИХ ФУНКЦІЙ ІЗ ВИКОРИСТАННЯМ ОПЕРАТОРІВ ВІДНОВЛЕННЯ ЗА ДАНИМИ НА ЛІНІЯХ

У сучасному математичному моделюванні фізичних та технічних процесів актуальною є проблема обробки й аналізу функцій багатьох змінних, значення яких відомі на системах ліній. Не виключенням є і задачі цифрової обробки зображень, зокрема чисельного інтегрування функцій з осциляцією. У задачах цифрової обробки зображень значна частина інформації про досліджуваний об’єкт надходить у вигляді вимірювань уздовж окремих напрямів або ліній, що є характерним для томографічних методів, дистанційного зондування та систем візу-

алізації. Розробка та застосування ефективних методів чисельного інтегрування швидкоосцильованих функцій на основі даних на системі ліній є важливою передумовою підвищення точності реконструкції, фільтрації та аналізу цифрових зображень.

Дослідження в статті присвячено чисельному інтегруванню щвидкоосцильованих функцій декількох змінних. Наведено кубатурну формули наближеного обчислення подвійних інтегралів від осцильованої експоненти. Кубатурна формула в своїй побудові в якості даних про функцію використовує сліди на взаємно перпендикулярних лініях. На класі диференційовних функцій представлено оцінки похибки наближення.

В роботі багато уваги приділено тестуванню кубатурної формули наближеного обчислення подвійних інтегралів від осцильованої експоненти. Отримані результати дозволяють пояснити вибір параметрів, підтвердити теоретичні оцінки похибки. Чисельний експеримент набуває особливої значущості, оскільки є базою для аналізу та прогнозування поведінки методу у тривимірному випадку. Це пов'язано з тим, що зі збільшенням розмірності зростає обсяг необхідної інформації про підінтегральну функцію, стає складніша структура похибки та істотно збільшуються обчислювальні витрати.

В роботі представлено кубатурну формулу наближеного обчислення потрійних інтегралів від осцильованої експоненти. Значення про функцію надаються як сліди функції на системі взаємно перпендикулярних прямих. Отримано оцінку похибки наближення на класі диференційовних функцій.

Ключові слова: *математичне моделювання процесів, цифрова обробка зображень, чисельне інтегрування швидкоосцильованих функцій багатьох змінних, кубатурна формула, відновлення функцій на лініях.*

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